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The Optimal Consumption Function in a Brownian Model of Accumulation

Part B: Existence of Solutions of Boundary Value Problems

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SRC Discussion Paper No 25

November 2014

Systemic Risk Centre

Discussion Paper Series

ISSN 2054-538X

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Keywords: Consumption, capital accumulation, Brownian motion, optimisation, ordinary differential equations, boundary value problems

AMS(2010) subject classifications: 34B40, 34C12, 34C45, 49J15, 49K15

JEL classifications: D900, E130, O410

This paper is published as part of the Systemic Risk Centre's Discussion Paper Series. The support of the Economic and Social Research Council (ESRC) in funding the SRC is gratefully acknowledged [grant number ES/K002309/1].

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Published by
Systemic Risk Centre
The London School of Economics and Political Science
Houghton Street
London WC2A 2AE

The Optimal Consumption Function in a Brownian Model of Accumulation Part B: Existence of Solutions of Boundary Value Problems *

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Revised Edition, 2014

Abstract

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*An earlier version of this paper was circulated as Discussion Paper No. TE/96/310, June 1996, Theoretical Economics Workshop, The Suntory Centre, LSE. The present text incorporates revisions up to 2014. My thanks go to Carol Hewlett≡Carol Foldes for support and help with word processing, Adam Ostaszewski for encourangement and comments, Ian Buckley for producing numerical simulations of some boundary value problems and to Mina Moshkeri and Jane Pugh for preparing diagrams. The support of the Economic and Social Research Council (ESRC) in funding the SRC is gratefully acknowledged [grant number ES/K002309/1].

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Abbreviated Title: Optimal Consumption as a Boundary Value Problem, Part B.

Introduction to Part B

As stated in the Abstract, this paper follows on from Foldes [1996] — hereinafter 'Part A' or simply [A] — comprising Sections 1 and 2 of our study. The present Introduction restates the boundary value problems (b.v.p.s) and certain assumptions formulated in Part A. This is followed by Sections 3–4, which are devoted to analysis of these problems and proof of the existence of solutions, with little further reference to the economic and probabilistic background.

We consider the following non-linear, non-autonomous system $S = (F, G)$ of o.d.e.s

$$
\begin{aligned}
h' &= F(h, \theta, z) = bh^2 + (2/\sigma^2)h[\theta - n + m/b - \frac{1}{2}b\sigma^2 - A] - 2[m - M]/b\sigma^2 \\
\theta' &= G(h, \theta, z) = (h - 1)\theta
\end{aligned}
\tag{0.1}
$$

where $h' = dh(z)/dz$, $\theta' = d\theta(z)/dz$, $A = A(z)$, $M = M(z)$, defined for $z \in \Re$ and points $\pi = (h, \theta)$ in a suitable domain $\mathbf{U} \subseteq \Re^2$. Recall that z stands for log-capital and $h(z) = dH(z)/dz$, $\theta(z) = \exp\{H(z) - z\}$, where $H(z)$ is log-consumption (both capital and consumption being measured in 'intensive' units). Here $b > 0$ is a coefficient of relative risk aversion — see [A](1.5) and (1.30) — and $\sigma^2 > 0$, n, m are 'compound' parameters which are defined in terms of the 'primitive' parameters of the stochastic growth model specified in [A], namely b and the means and variances μ_η, σ_η^2, $\eta = \alpha, \beta, \gamma, \rho$ of the four Brownian motions, see [A](1.7) et seq. The functions A and M are defined for $z \in \Re$ in terms of the 'intensive' production function ψ by

$$
A(z) = \psi(\kappa)/\kappa = a(\kappa), \quad M(z) = \psi'(\kappa) = d\psi(\kappa)/d\kappa, \quad z = \ln \kappa, \quad \kappa > 0,
\tag{0.2}
$$

and are (at least) $\mathbf{C}^1$ with one-sided limits

$$
A(-\infty) = M(-\infty) = \psi'(0) = \psi_0', \quad A(\infty) = M(\infty) = \psi'(\infty) = 0.
\tag{0.3}
$$

Recall that ψ is defined for $\kappa \geq 0$ with $\psi(0) = 0$, and is (at least) $\mathbf{C}^2$ with $\psi'(\kappa) > 0 > \psi''(\kappa)$ for $0 < \kappa < \infty$ and limits

$$
0 < \psi_0' = \psi'(0) < \infty, \quad \psi'(\infty) = 0.
\tag{0.4}
$$

We further introduce the following constants:

$$N = n + (b-1)\psi_0'/b \tag{0.5}$$

$$q = n + (b-1)(m + \frac{1}{2}b\sigma^2)/b \tag{0.6}$$

$$Q = n - m/b + \frac{1}{2}b\sigma^2 = q - m + \frac{1}{2}\sigma^2, \tag{0.7}$$

cf.[A](1.13–16) and (1.40). In line with the statements of Theorems 2 and 3 of Part A, we adopt throughout, without special mention, the following

STANDING ASSUMPTIONS.

(0.8) If $b > 1$, then $N > 0$ and {either $n > 0$ or $q > 0$}.

(0.9) If $b < 1$, then $n > 0$ and {either $N > 0$ or $q > 0$}.

(0.10) If $b = 1$, then $N = n = q > 0$.

Usually we leave aside without special mention cases with $n = 0$ or $N = 0$, regarding which see fns. 3 and 6 and Fig. 5 below.

The main object of this Part is to show that each of the b.v.p.s defined by the statements of Theorems 2 and 3 in [A] has a solution. The following theorem is based on these statements, but a more precise version will be given later.

THEOREM 4A (Existence of Solutions of b.v.p.s).

In each of the following cases the system $S = (F, G)$ defined by (0.1) has a solution $\phi^* = (h^*, \theta^*) = (h^*(z), \theta^*(z))$ which is defined for all $z \in \Re$ and converges for $z \to \pm\infty$ to limits satisfying the following conditions:

'Type 1' b.v.p.s. (cf. Theorem 2): If $b > 0$, $n > 0$ and $N > 0$, the limits are

$$h^*(+\infty) = 1, \quad \theta^*(+\infty) = n. \tag{0.11}$$

$$h^*(-\infty) = 1, \quad \theta^*(-\infty) = N. \tag{0.12}$$

'Type 0' b.v.p.s. (cf. Theorem 3):

(i) If $b > 1$, $N > 0$ and $q > 0 \geq n$, the limits are (0.12) and $(h^*(+\infty), 0)$ for some $h^*(+\infty)$ satisfying

$$1/b < h^*(+\infty) \leq 1. \tag{0.13}$$

(ii) If $b < 1$, $n > 0$ and $q > 0 \geq N$, the limits are (0.11) and $(h^*(-\infty), 0)$ for some $h^*(-\infty)$ satisfying

$$1/b > h^*(-\infty) \geq 1. \tag{0.14}$$

We call a solution $\phi^* = (h^*, \theta^*)$ of S which satisfies one of the sets of conditions of this theorem a solution of 'the' (appropriate) b.v.p. — of Type 1, 0(i) or 0(ii) — or simply a 'star' solution. Taking into account the results of Part A, *a proof that a star solution exists is a proof that the underlying model of accumulation admits an optimal log-consumption function* $H(z)$, where $H'(z) = h(z)$ and $\theta(z) = \exp\{H(z) - z\}$. While our main aim will be to prove the existence and uniqueness of star solutions, we shall also consider the properties of solutions of S generally. Apart from any mathematical interest which this rather unusual system of o.d.e.s may possess, it is useful to have some insight into the economic consequences of choosing the 'wrong' solution as the consumption function.

3 Phase Analysis

(i) *Generalities.*

The present Section gives a preliminary discussion of S followed by a detailed discussion of certain auxiliary systems which define bounds for the motion of S; further details about S are established in Section 4.

To begin with, a brief survey of some properties of S. It is necessary to bear in mind that the independent variable is not time but log-capital, but we shall nevertheless slip into much of the usual terminology of forward ($z \uparrow$) and backward ($z \downarrow$) motion and limits, stable/unstable or in/out curves which move towards/away from a given point for the *forward* motion, etc. In the Figures, phase arrows always refer to the 'forward' motion. A point of the plane $\Re^2$ is often written $\pi = (h, \theta)$. Plane sets are written with curly brackets, often omitting the argument $\pi = (h, \theta)$, e.g. $\{\theta > 0\} = \{(h, \theta) \colon h \in \Re, \theta > 0\}$, $\{\theta \geq 0\} = \{(h, \theta), h \in \Re, \theta \geq 0\}$. Occasionally, sets in $\Re^3$ are written with bold curly brackets $\boldsymbol{\{} \cdots \boldsymbol{\}}$. Given a suitable domain $\mathbf{U} \subseteq \Re^2$ and an interval $I \subseteq \Re$, we say that a *solution* of S (on I) is a function $z \mapsto \phi(z) = (h(z), \theta(z))$ of class $\mathbf{C}^1$ from I into $\mathbf{U}$, satisfying (0.1) for $z \in I$. The corresponding curve in the (h, θ) plane $\Re^2$ — i.e. the image set $\{\phi(z); z \in I\}$ parametrised and ordered by I — is called the solution path or simply the *path* of ϕ on I. A solution is called *bounded* on an interval if both $h(z)$ and $\theta(z)$ are bounded there, otherwise it is *unbounded.* We consider phase diagrams (or rather path diagrams) with θ on the horizontal axis and h on the vertical; thus we speak of motion to the left or right, up or down. Unless otherwise stated or implied, we choose as the domain $\mathbf{U}$ for S either $\{(h, \theta) \colon h \in \Re, \theta > 0\} = \{\theta > 0\}$ or $\{(h, \theta) \colon h \in \Re, \theta \geq 0\} = \{\theta \geq 0\}$, or occasionally the whole (h, θ) plane; the choice of domain will usually be clear from the context. Note that the axis $\{\theta = 0\}$ acts as a barrier to left/right motion, so that a solution path must lie entirely in one of the sets $\{\theta > 0\}$, $\{\theta = 0\}$ or $\{\theta < 0\}$.[1]

[1]Formally, the domain $\mathbf{U}$ is considered as a metric space 'in itself', with the 'relative' topology and metric induced in $\mathbf{U}$ as a subset of $\Re^2$. Thus, if $\mathbf{U} = \{\theta > 0\}$, 'open' and 'relatively open' are equivalent for subsets of $\{\theta > 0\}$, but a relatively closed set may not be closed in the plane; in this case, the relative closure of a set $\mathcal{A}$ is written $[\![\mathcal{A}]\!]$.

If $\mathbf{U} = \{\theta \geq 0\}$, 'closed' and 'relatively closed' are equivalent for subsets of $\{\theta \geq 0\}$, while a relatively open set is of the form $N \cap \{\theta \geq 0\}$, where N is open in $\Re^2$. Given a point $(h_0, 0)$ in the boundary $\{\theta = 0\}$, convergence $(h_n, \theta_n) \to (h_0, 0)$ is defined in terms of sequences with $\theta_n \geq 0$, so that derivatives evaluated at such a point are, strictly speaking, 'one-sided' limits, and results about Jacobians, eigenvalues etc., are to be interpreted accordingly; however it will usually not be necessary to insist on such distinctions.

Analogous remarks apply to other choices of $\mathbf{U}$, e.g. $\{h > 0, \theta > 0\}$ or $\{h \geq 0, \theta \geq 0\}$.

Since the functions F and G are $\mathbf{C}^1$ in (h,θ,z), a unique local solution 'through' a given point $(\pi_\diamondsuit, z_\diamondsuit) = (h_\diamondsuit, \theta_\diamondsuit, z_\diamondsuit)$ — i.e. a solution $\phi(z)$ satisfying $\phi(z_\diamondsuit) = \pi_\diamondsuit$ — always exists and may be continued on a maximal interval $I(\pi_\diamondsuit, z_\diamondsuit) = (z_-(\pi_\diamondsuit, z_\diamondsuit), z_+(\pi_\diamondsuit, z_\diamondsuit))$ or (z_-, z_+) for short; its position at $z \in I(\pi_\diamondsuit, z_\diamondsuit)$ may be written $\phi(z;\pi_\diamondsuit, z_\diamondsuit) = (h(z;\pi_\diamondsuit z_\diamondsuit), \theta(z;\pi_\diamondsuit, z_\diamondsuit))$. If we consider the solution only for $z \geq z_\diamondsuit$, or only for $z \leq z_\diamondsuit$, we sometimes write $\phi^\triangleright$ or $\phi^\triangleleft$ in place of ϕ, $I^\triangleright = [z_\diamondsuit, z_+)$ or $I^\triangleleft = (z_-, z_\diamondsuit]$ in place of I, and refer to the forward or backward solution through $(\pi_\diamondsuit, z_\diamondsuit)$; also we often assume without special mention that the solution is continued to the whole of $I^\triangleright$ or $I^\triangleleft$. We call $(\pi_\diamondsuit, z_\diamondsuit)$ (or just $\pi_\diamondsuit$ or $z_\diamondsuit$) the *start* of the solution.

According to standard results, a solution whose path stays bounded (uniformly in z) as $z \uparrow$ ($z \downarrow$), can be continued to $z_+ = \infty$ ($z_- = -\infty$), see Nemytskii and Stepanov [1960] T.1.21 and T.1.34, also Coddington and Levinson [1955], Ch.2, problems 4 and 5. Here we can do slightly better, as follows:

PROPOSITION 1. If, for a given solution $\phi = (h,\theta)$ of S with $\theta \geq 0$, $h(z)$ stays bounded as $z \uparrow$ ($z \downarrow$), then the solution can be continued to $z_+ = \infty$ ($z_- = -\infty$).

PROOF. Suppose that $\phi(z) = (h(z), \theta(z))$ is defined for $z \in [z_\diamondsuit, z_+)$ with $\phi(z_\diamondsuit) = \pi_\diamondsuit = (h_\diamondsuit, \theta_\diamondsuit)$ and that $|h(z)| \leq \alpha$ for $z \in [z_\diamondsuit, z_+)$. Let $\theta_\diamondsuit \geq 0$ and suppose that $z_+ < \infty$. Now $\theta' = (h-1)\theta$ implies $0 \leq \theta(z) \leq \theta_\diamondsuit \cdot \exp\{\alpha(z_+ - z_\diamondsuit)\}$ for $z \in [z_\diamondsuit, z_+)$. But then $\theta(z_+)$ exists as a finite limit. Applying the preceding inequality together with $|h(z)| \leq \alpha$ to the equation $h' = F(h,\theta,z)$ it is found that $h(z_+)$ also exists as a finite limit, and the usual continuation argument shows that the solution can be continued forward from $(h(z_+), \theta(z_+), z_+)$, contrary to the assumption that $z_+ < \infty$. The argument for $z \downarrow z_-$ is analogous. $\|$

A related question concerns the continuation of *paths.* It follows from the equation $\theta' = (h-1)\theta$ that the (forward) motion of S is always to the left ($\theta \downarrow$) in the region $\{h < 1, \theta > 0\}$, always to the right ($\theta \uparrow$) in the region $\{h > 1, \theta > 0\}$. Consequently, given the solution $(h(z;\pi_\diamondsuit, z_\diamondsuit), \theta(z;\pi_\diamondsuit, z_\diamondsuit))$ through a point $(\pi_\diamondsuit, z_\diamondsuit) = (h_\diamondsuit, \theta_\diamondsuit, z_\diamondsuit)$ with $\theta_\diamondsuit > 0$ and $h_\diamondsuit \neq 1$, we may take θ as path parameter, i.e. we may represent the path locally as the solution $h = h(\theta;\pi_\diamondsuit, z_\diamondsuit)$ of the equation $dh/d\theta = F(h,\theta,z(\theta))/(h-1)\theta$, where $z(\theta)$ is the function inverse to the monotone function $\theta(z;\pi_\diamondsuit, z_\diamondsuit)$. *This representation can be continued as $\theta \downarrow$ and as $\theta \uparrow$ so long as $h(z;\pi_\diamondsuit, z_\diamondsuit) - 1$ is defined and finite and keeps the same (definite) sign.* In particular, *no path can terminate in the interior of either of the regions $\{h < 1, \theta > 0\}$ or $\{h > 1, \theta > 0\}$ as $z \uparrow z_+$ or as $z \downarrow z_-$.*

Difficulties with the system S arise from its being non-linear, non-autonomous, with no stationary point, incomplete (i.e. finite escape levels z_+ or z_- occur), and unstable with respect to perturbation of initial values. This instability is present in particular along paths which converge to one of the boundary values prescribed in Theorem 4A; in fact, the system possesses a version of the 'knife-edge' property found in certain deterministic models of economic growth. To set against these vices there are virtues. All solutions *converge to limits*, finite or infinite, at the endpoints of their intervals of definition. The path map has some simplifying features. Motion is always to the left if $h < 1$, $\theta > 0$, to the right if $h > 1$, $\theta > 0$. There are also *order-preserving properties*: loosely speaking, if $(h^i_\diamond, \theta^i_\diamond, z_\diamondsuit), i = 0, 1$ are distinct points, an inequality of the form $0 < h^1 < h^0, 0 < \theta^1 < \theta^0$ is preserved along solutions through these points as $z \uparrow$ (at least while both solutions remain defined with both co-ordinates positive), while an inequality $0 < h^1 < h^0, 0 < \theta^0 < \theta^1$ is preserved as $z \downarrow$ (with the same proviso). Closely connected with these ordering properties, it is possible to define autonomous systems which give simple *upper and lower bounds* for the motion of S (with more accurate bounds if only large $|z|$ are considered). Most important, the system is *asymptotically autonomous* for both $z \to \infty$ and $z \to -\infty$; we begin our detailed discussion with this last point.

Referring to (0.1) and the definitions of the functions M and A, it is seen that, for given (h, θ), the term $(2/\sigma^2)[M(z)/b - hA(z)]$ tends to zero as $z \to \infty$ and to $(2/\sigma^2)\psi_0'[1/b - h]$ as $z \to -\infty$, so that the function $F(h, \theta, z)$ converges, uniformly on (h, θ)-compacts of $\Re^2$, to the functions

$$\begin{aligned}
(3.1)\qquad F_\infty(h,\theta) &= bh^2 + (2/\sigma^2)h[\theta - n + m/b - \frac{1}{2}b\sigma^2] - 2m/b\sigma^2 \\
&= F(h,\theta,z) - (2/\sigma^2)[M/b - hA], \\
(3.2)\qquad F_{-\infty}(h,\theta) &= bh^2 + (2/\sigma^2)h[\theta - N + (m-\psi_0')/b - \frac{1}{2}b\sigma^2] - 2(m-\psi_0')/b\sigma^2 \\
&= F(h,\theta,z) - (2/\sigma^2)[(M-\psi_0')/b - h(A-\psi_0')] \\
&= F_\infty(h,\theta) - (2\psi_0'/\sigma^2)(h - 1/b)
\end{aligned}$$

as $z \to \infty$ and $z \to -\infty$ respectively, taking into account that $N = n + (b-1)\psi_0'/b$. Since $G = (h-1)\theta$ does not depend on z, it follows that the system $S = (F, G)$ is asymptotically autonomous in the sense of Markus [1956], with limiting 'autonomous systems at $\pm\infty$' defined by $S_\infty = (F_\infty, G)$ and $S_{-\infty} = (F_{-\infty}, G)$; see also Opial [1960]. Some simple but useful properties follow immediately; we state them for the forward motion only.

PROPOSITION 2. (i) If a solution ϕ of S is defined on a maximal forward interval $I^\triangleright = [z_\Diamond, z_+)$ and the forward limit set[2] $\Pi^\triangleright$ of ϕ is not empty, then $\Pi^\triangleright$ is the union of paths of S_∞. If ϕ is bounded on $I^\triangleright$, then $\Pi^\triangleright$ is compact and non-empty.

(ii) Consequently, if ϕ *converges to a point* $\pi^\triangleright = (h^\triangleright, \theta^\triangleright)$, then that point must be a stationary point of S_∞.

(iii) If $\pi^\triangleright$ is a stationary point of S_∞ and the variational equations of this system based on $\pi^\triangleright$ have eigenvalues with negative real parts, then there is a neighbourhood $\mathcal{N}$ of $\pi^\triangleright$ and a number $z_\Diamond$ such that every solution of S whose path meets $\mathcal{N}$ at some $z > z_\Diamond$ converges to $\pi^\triangleright$.

(iv) If $\mathcal{G}$ is an unbounded open region of the (h, θ)-plane from which paths of S and of S_∞ do not escape, and if all paths of S_∞ which enter $\mathcal{G}$ become unbounded, then the same is true of paths of S.

These results allow information about solutions of S to be obtained from corresponding information about the asymptotic systems, whose phase picture is relatively simple. In particular it will be found that, for each combination of parameters considered in Theorem 4A, each of the asymptotic systems has at most three stationary points in the half-plane $\{\theta \geq 0\}$, one of which is a saddle while the others are stable or unstable nodes.[3] According to property (ii) above, a star solution must converge at each end to one of these points. It turns out that the co-ordinates of the saddles, and only these, satisfy the prescribed conditions. *The problem of proving that a particular b.v.p. has a solution is therefore equivalent to proving the existence of a sort of saddle connection* (but between saddles of the asymptotic systems, not of S).

This way of stating the matter suggests an analysis designed to show that there is a pair of two-dimensional manifolds of solutions of S, converging respectively to the saddle points of S_∞ ($S_{-\infty}$) as $z \uparrow$ ($z \downarrow$), which intersect transversely in a single curve defining a star solution. This is essentially what we shall do, but in a way which relies as much as possible on elementary methods using phase analysis in the plane.[4]

[2]A point $\pi^\triangleright = (h^\triangleright, \theta^\triangleright)$ belongs to the forward (or 'omega') limit set of ϕ if there is a sequence (z_k), converging to z_+ as $k \to \infty$, such that $h(z_k) \to h^\triangleright$, $\theta(z_k) \to \theta^\triangleright$.

[3]There is a minor qualification in the cases $0 = n < q$ or $0 = N < q$, where there is a saddle-node bifurcation; see below, fn. 6 and Fig. 5.

[4]It is possible to imbed S in an autonomous three-dimensional system $\mathfrak{S}$ of class $\mathbf{C}^1$ in such a way that the stationary points (in particular the saddles) of the asymptotic systems correspond to stationary points (saddles) of $\mathfrak{S}$. It is intended to pursue this approach in Part C.

In addition to the systems $S_{\pm\infty} = (F_{\pm\infty}, G)$, we shall need to consider certain other auxiliary two-dimensional autonomous systems which will serve to define bounds for the motion of S. The rest of Section 3 is concerned with these systems. In order to establish a unified notation we write

$$\bar{F} = \bar{F}(h,\theta) = bh^2 + (2/\sigma^2)h(\theta - \bar{Q}) - 2\bar{m}/b\sigma^2. \tag{3.3a}$$

Thus (for fixed σ^2) a triple of parameters $(b, \bar{Q}, \bar{m})$ satisfying suitable conditions defines a system

$$\bar{S} = (\bar{F}, G)\colon h' = \bar{F}(h,\theta), \quad \theta' = G(h,\theta) = (h-1)\theta. \tag{3.3b}$$

We label parameters according to the systems to which they belong.

Other parameters of importance are the numbers

$$\bar{\theta}_1 \text{ defined as the solution of } \bar{F}(1,\theta) = 0, \tag{3.4}$$

$$\bar{\theta}_{1/b} \text{ defined as the solution of } \bar{F}(1/b,\theta) = 0, \tag{3.5}$$

$$\bar{R} \doteq -\bar{F}(0,\theta) = 2\bar{m}/b\sigma^2. \tag{3.6}$$

Then we have, from (3a),

$$\bar{\theta}_1 = \bar{Q} - \frac{1}{2}b\sigma^2 + \bar{m}/b, \tag{3.4a}$$

$$\bar{\theta}_{1/b} = \bar{Q} - \frac{1}{2}\sigma^2 + \bar{m}. \tag{3.5a}$$

Thus a system $\bar{S}$ can also be defined (for fixed σ^2) by specifying b, $\bar{m}$ and either $\bar{\theta}_1$ or $\bar{\theta}_{1/b}$.

We shall consider only systems $\bar{S}$ for which either $\bar{\theta}_1 > 0$, called Type 1 Systems, or $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, called Type 0 Systems. Usually we leave aside, without special mention, systems with $\bar{\theta}_1 = 0$. If $b = 1$, then $\bar{\theta}_1 = \bar{\theta}_{1/b} > 0$, so that only Type 1 systems occur. Note that, for a Type 0 System, $\bar{\theta}_{1/b} > 0 > \bar{\theta}_1$, and (4a)–(5a) imply

$$\textit{either } \{b > 1 \text{ and } \bar{m} + \frac{1}{2}b\sigma^2 > 0\} \textit{ or } \{0 < b < 1 \text{ and } \bar{m} + \frac{1}{2}b\sigma^2 < 0\}. \tag{3.6a}$$

In case $\bar{m} = 0$. $\bar{Q} > 0$ whichever the Type, and Type 0 can occur only if $b > 1$. For brevity, we often omit special discussion of cases with $b = 1$ and those with $\bar{m} = 0$.

To define $\bar{F}(h,\theta)$ so that it coincides with $F(h,\theta,\bar{z})$ for a *fixed* $\bar{z} \in [-\infty,\infty]$, set

(3.6b) $$\bar{Q} = Q + \bar{A}, \quad \bar{m} = m - \bar{M}, \text{ where } \bar{A} = A(\bar{z}), \quad \bar{M} = M(\bar{z}),$$

cf.(0.1–0.3); then, using (0.7), we have

(3.6c) $$\bar{\theta}_1 = n + \bar{A} - \bar{M}/b, \quad \bar{\theta}_{1/b} = q + \bar{A} - \bar{M}.$$

In particular, using (0.3–0.6) we obtain for $\bar{F} = F_\infty$

(3.7) $$\bar{Q} = Q,\ \bar{m} = m,\ \bar{\theta}_1 = n,\ \bar{\theta}_{1/b} = q,\ \bar{R} = R_\infty = 2m/b\sigma^2;$$

and for $\bar{F} = F_{-\infty}$

(3.8) $$\begin{aligned} &\bar{Q} = Q + \psi_0',\ \bar{m} = m - \psi_0',\ \bar{\theta}_1 = N,\ \bar{\theta}_{1/b} = q,\\ &\bar{R} = R_{-\infty} = 2(m - \psi_0')/b\sigma^2. \end{aligned}$$

Thus S_∞ ($S_{-\infty}$) is a Type 1 system if $n > 0$ ($N > 0$); a Type 0 system if $q > 0 \geq n$ ($q > 0 \geq N$).

A word here about classification and terminology. We have said previously that a *b.v.p.* of Type 1 arises if $b > 0$ and both $n > 0$ and $N > 0$, i.e. if both S_∞ and $S_{-\infty}$ are Type 1 *Systems.* A *b.v.p.* of Type 0(i) arises if $b > 1$, $N > 0$ and $q > 0 \geq n$, i.e. if $b > 1$, $S_{-\infty}$ is Type 1 and S_∞ is Type 0; again, a b.v.p. of Type 0(ii) arises if $b < 1$, $n > 0$ and $q > 0 \geq N$, i.e. if $b < 1$, S_∞ is Type 1 and $S_{-\infty}$ is Type 0. *Thus the Standing Assumptions require that at least one of $S_{-\infty}$ and S_∞ be a Type 1 System.* Type 1 b.v.p.s are those for which this holds for both systems; Type 0 b.v.p.s are those for which only one of the systems is Type 0, and then the sign of $b-1$ defines the case. These remarks define the main criteria according to which both auxiliary systems and b.v.p.s will be classified: first as Type 1 or 0, then according to the sign of $b - 1$, (which affects the analysis of b.v.p.s of both Types). The sign of $\bar{m}$ then defines a further criterion for individual systems $\bar{S}$, while for b.v.p.s there is a classification according to the signs of m and $m - \psi_0'$ (yielding three cases if the borderline values $m = 0$ and $m = \psi_0'$ are left aside, which we shall sometimes do). This classification is reflected in Figures 2–4, which are explained below.

(ii) *Solutions, Phase Contours and Stationary Points of Systems* $\bar{S} = (\bar{F}, G)$.

Definitions for systems $\bar{S}$ concerning domain, solutions and phase maps will be analogous to those for S, allowing for the autonomy of $\bar{S}$. These will not be set out in full, but some details are given in the footnote below.[5]

We begin by examining the contours (level curves) of the functions G and $\bar{F}$ in the (h, θ) plane. Evidently the contours of $G = (h-1)\theta$ are rectangular hyperbolae with asymptotes $\theta = 0$ and $h = 1$; we shall not stop to draw these. In the half-plane $\{\theta > 0\}$ we have $\theta' = G > 0$ when $h > 1$ and $\theta' = G < 0$ when $h < 1$. *Further, since* $\theta' = 0$ *only along* $\{h = 1\}$ *and* $\{\theta = 0\}$, *any stationary point of* $(\bar{F}, G)$ *must lie on one of these lines.*

Referring next to (3) and (6), we note that the equation $\bar{F}(h, \theta) = \gamma$, where γ is a constant, may be rewritten as

$$(3.9) \qquad \frac{1}{b}(\bar{F} - \gamma) = [h + (\theta - \bar{Q})/b\sigma^2]^2 - [(\theta - \bar{Q})/b\sigma^2]^2 - (\bar{R} + \gamma)/b = 0.$$

For $\gamma \neq -\bar{R}$, this is the equation of a hyperbola with centre at $(h = 0, \theta = \bar{Q})$, axes $h = -(\theta - \bar{Q})/b\sigma^2$ and $\theta = \bar{Q}$, and asymptotes $h = -2(\theta - \bar{Q})/b\sigma^2$ and $h = 0$. The hyperbola consists of two distinct curves, one in $\{h > 0\}$ the other in $\{h < 0\}$, which we call the positive and negative contours of $\bar{F}$ at the level γ and denote by $\bar{F}^+(\gamma)$ and $\bar{F}^-(\gamma)$. In case $\gamma = -\bar{R}$, the contours are the asymptotes. See Figure 1. It is clear that the asymptotes define the boundaries of four domains, with contours at level $\gamma > -\bar{R}$ to the 'north-east' and 'south-west' and those for $\gamma < -\bar{R}$ to the 'north-west' and 'south-east'. The slope $s(h, \theta) = dh/d\theta$ along $\bar{F} = \gamma$ (at a point (h, θ) different from the centre

[5] Given a domain $\mathbf{U}$ and an interval $I \subseteq \Re$, a solution of $\bar{S}$ on I is a function $z \to \bar{\phi}(z) = (\bar{h}(z), \bar{\theta}(z))$ of class $\mathbf{C}^1$ satisfying (3.3b) for $z \in I$. Its *path* is the (directed) curve $\{\bar{\phi}(z), z \in I\}$ in $\Re^2$. Unless otherwise stated or implied, the domain for $\bar{S}$ will be $\{\theta \geq 0\} = \{h \in \Re, \theta \geq 0\}$. Given a pair $(\pi_\diamondsuit, z_\diamondsuit) \in \mathbf{U} \times \Re$, a local solution $\bar{\phi}(z; \pi_\diamondsuit, z_\diamondsuit)$ satisfying $\bar{\phi}(z; \pi_\diamondsuit, z_\diamondsuit) = \pi_\diamondsuit$ exists and may be continued on a maximal open interval $I(\pi_\diamondsuit, z_\diamondsuit) = (z_-(\pi_\diamondsuit, z_\diamondsuit), z_+(\pi_\diamondsuit, z_\diamondsuit))$ containing $z_\diamondsuit$. If $z_\diamondsuit = 0$, write this solution as $\bar{\phi}^0(z; \pi_\diamondsuit)$ or simply $\bar{\phi}^0_z \pi_\diamondsuit$, so that $\bar{\phi}^0_0 \pi_\diamondsuit = \pi_\diamondsuit$, and write the interval of definition as $I^0(\pi_\diamondsuit)$. Then, by autonomy of $\bar{S}$,

$$\bar{\phi}^0(\zeta; \pi_\diamondsuit) = \bar{\phi}(z; \pi_\diamondsuit, z_\diamondsuit) \quad \text{where } \zeta = z - z_\diamondsuit \in I^0(\pi_\diamondsuit) = I(\pi_\diamondsuit, z_\diamondsuit) - z_\diamondsuit.$$

Further, using abridged notation, if

$$z \in I^0(\pi_0), \; \pi_1 = \bar{\phi}^0_z \pi_0, \text{ and } s \in I^0(\pi_1), \; \pi_2 = \phi^0_s \pi_1$$

then $s \in I^0(\pi_0)$ and $\bar{\phi}^0_{z+s}\pi_0 = \bar{\phi}^0_s \pi_1 = \bar{\phi}^0_s \circ \bar{\phi}^0_z \pi_0$. (Group property for the local flow $\bar{\phi}^0$ of $\bar{S}$.) If only forward (or only backward) motion is considered, replace $I(\pi_\diamondsuit, z_\diamondsuit)$ by $I^\triangleright(\pi_\diamondsuit, z_\diamondsuit) = [z_\diamondsuit, z_+(\pi_\diamondsuit, z_\diamondsuit))$, $I^0(\pi_\diamondsuit)$ by $I^{\triangleright 0}(\pi_\diamondsuit) = [0, z_+(\pi_\diamondsuit, 0))$ to obtain a semigroup property for the forward local semiflow $\bar{\phi}^0$; similarly for the backward motion. Cf. Hirsch [1984], p.27.

of the hyperbola) is given by

$$s(h,\theta) = -(\partial\bar{F}/\partial\theta)/(\partial\bar{F}/\partial h) = -h/(bh\sigma^2 + \theta - \bar{Q}). \tag{3.9a}$$

We have $\partial\bar{F}/\partial h = 0$ along the first axis (leaving aside the centre), so that the contours have vertical slope there. Also $\bar{F} = -\bar{R}$ along both asymptotes, so that in particular the motion on the horizontal axis is up/down according as $\bar{R}$ (or $\bar{m}$) is $-/+$.

Contours for $\gamma > -\bar{R}$ have negative slope throughout and both contours cross every vertical line, so that the quadratic equation $\bar{F}(h,0) = \gamma$ has two real solutions h of opposing signs. For $\gamma = -\bar{R}$, one solution is positive, one zero. For $\gamma < -\bar{R}$, the two contours lie on opposite sides of a certain vertical open strip, so that no real solutions exist if the line $\{\theta = 0\}$ lies in this strip; however, if $\bar{F}^+(\gamma)$ meets $\{\theta > 0\}$, there are two distinct positive solutions. We denote the real solutions of $\bar{F}(h,0) = 0$ (when they exist) by $\bar{h}^+$ and $\bar{h}^-$; thus

$$b\sigma^2\bar{h}^{\pm} = \bar{Q} \pm [\bar{Q}^2 + 2\bar{m}\sigma^2]^{\frac{1}{2}}. \tag{3.10}$$

For $\gamma = 0$, these remarks yield the following consequences:

PROPOSITION 3. $\bar{h}^+ > 0 > \bar{h}^-$ iff $\bar{R} > 0$; $\bar{h}^+ > 0 = \bar{h}^-$ iff $\bar{R} = 0$;
$\bar{h}^+ > \bar{h}^- > 0$ iff $\bar{R} < 0$ and $\bar{F}^+(0)$ meets $\{\theta > 0\}$.

Referring now to the definition (4) of $\bar{\theta}_1$, we note that, if $\bar{\theta}_1 > 0$, the contour $\bar{F}^+(0)$ must cut $\{\theta = 0\}$ at a point $\bar{h}^+ > 1$. If in addition $\bar{R} < 0$, there is a second intersection at $\bar{h}^-$ with $\bar{h}^- > 0$, alternatively $\bar{h}^- \leq 0$ if $\bar{R} \geq 0$, and in either case $\bar{h}^+ > 1 > \bar{h}^-$. Again, by (5), if $\bar{\theta}_{1/b} > 0$, then $\bar{F}^+(0)$ must cut $\{\theta = 0\}$ at some $\bar{h}^+ > 1/b$. If in addition $\bar{R} < 0$, there is a second intersection at $\bar{h}^-$ with $\bar{h}^- > 0$, alternatively $\bar{h}^- \leq 0$ if $\bar{R} \geq 0$, and in either case $\bar{h}^+ > 1/b > \bar{h}^-$. *Thus the assumption that $\bar{\theta}_1 \vee \bar{\theta}_{1/b} > 0$ ensures that there are always distinct real solutions $\bar{h}^+$ and $\bar{h}^-$ of $\bar{F}(h,0) = 0$.* Further inequalities which are easily checked from diagrams are set out in the following

PROPOSITION 4. Distinct real solutions $\bar{h}^+$ and $\bar{h}^-$ of the equation $\bar{F}(h,0) = 0$ exist in all cases if $\bar{\theta}_1 \vee \bar{\theta}_{1/b} > 0$, with $\bar{h}^+ > 0$ and sgn $(\bar{h}^-) =$ sgn $(-\bar{R})$.

If $\bar{\theta}_1 > 0$, then $\bar{h}^+ > 1 > \bar{h}^-$.

If $\bar\theta_{1/b} > 0$, then $\bar h^+ > 1/b > \bar h^-$.

If $\bar\theta_1 > 0$ and $b > 1$, then sgn $(\bar\theta_{1/b}) =$ sgn $(1/b - \bar h^-)$.

If $\bar\theta_1 > 0$ and $b < 1$, then sgn $(\bar\theta_{1/b}) =$ sgn $(\bar h^+ - 1/b)$.

If $\bar\theta_{1/b} > 0 \geq \bar\theta_1$ and $b > 1$, then $1 \geq \bar h^+ > 1/b > \bar h^-$, and $\bar h^+ = 1$ only if $\bar\theta_1 = 0$.

If $\bar\theta_{1/b} > 0 \geq \bar\theta_1$ and $b < 1$, then $\bar h^+ > 1/b > \bar h^- \geq 1$, and $\bar h^- = 1$ only if $\bar\theta_1 = 0$.

These assertions apply in particular to $\bar F = F_\infty$ or $\bar F = F_{-\infty}$, with parameter values as in (7–8). In these cases we write $\bar h^\pm$ as $h^\pm_\infty$ or $h^\pm_{-\infty}$.

Since a stationary point of $\bar S$ must satisfy $\bar F = G = 0$, (and we consider only points with $\theta \geq 0$), it follows that Type 1 systems (those with $\bar\theta_1 > 0$) have precisely three such points, namely

$$(1, \bar\theta_1), \qquad (\bar h^+, 0), \qquad (\bar h^-, 0),$$

while Type 0 systems (those with $\bar\theta_{1/b} > 0 \geq \bar\theta_1$) have only two, namely

$$(\bar h^+, 0) \qquad \text{and} \qquad (\bar h^-, 0).$$

The results of the geometric discussion so far are illustrated in Figures 2, which show both Type 1 and Type 0 systems $\bar S$ with $\bar m < 0$ and $\bar m > 0$, distinguishing between Type 0 systems with $b > 1$ and those with $b < 1$. (There is no Fig. 2(vi), because by (6a) the conditions $b < 1$, $\bar\theta_{1/b} > 0 \geq \bar\theta_1$, $\bar m \geq 0$ are inconsistent). It is clear enough from the Figures that in Type 1 systems the point $(1, \bar\theta_1)$ is a saddle, $(\bar h^+, 0)$ is an unstable node and $(\bar h^-, 0)$ is a stable node, but some calculations are also given below. Again, in Type 0 systems with $b > 1$, the point $(\bar h^+, 0)$ is a saddle and $(\bar h^-, 0)$ is a stable node; while for Type 0 with $b < 1$, the point $(\bar h^+, 0)$ is an unstable node and $(\bar h^-, 0)$ is a saddle.[6] For each saddle point we have drawn in the 'stable' manifold (or curve) $\bar{\mathcal{M}}^\triangleright$ (labelled $\bar f$) and the 'unstable' manifold $\bar{\mathcal{M}}^\triangleleft$ (labelled $\bar g$), except that in Type 0 systems one of the manifolds lies on the vertical axis; more of these manifolds later.

In the particular cases $\bar F = F_{\pm\infty}$, information about phase behaviour is also shown in Figures 3–4. Fig. 3, comprising six diagrams, illustrates cases where both S_∞ and $S_{-\infty}$

[6]Once again, there is a qualification, relating to cases with $\theta_{1/b} > \theta_1 = 0$. There is a saddle-node bifurcation at $(1, \bar\theta_1) = (1, 0)$, but the phase behaviour remains saddle-like in the closed right half-plane and we treat the point as a saddle without special discussion. The stable node remains at $(\bar h^-, 0)$ if $b > 1$, the unstable node at $(\bar h^+, 0)$ if $b < 1$. Space does not allow a detailed discussion of this amusing case, but an example is illustrated in Fig. 5.

are of Type 1 (corresponding to Type 1 b.v.p.s), classified according to the sign of $b-1$ and then according to the signs of m and $m-\psi_0'$. Fig. 4 has the same information for cases where either $b>1$ and S_∞ is of Type 0 or $b<1$ and $S_{-\infty}$ is of Type 0 (except that there is no Fig. 4(vi), because by (6a) and (8) the conditions $b<1$, $q>0\geq N$, $m\geq\psi_0'$ are inconsistent). Each diagram shows (where space permits) the curves $F_\infty=0$ and $F_{-\infty}=0$ and the stable and unstable curves at saddle points of S_∞ and $S_{-\infty}$ (as well as other information to be explained later). In the sequel, the saddle points of $\bar{S}$, S_∞ and $S_{-\infty}$ (of whichever Types), will be denoted $\bar{\pi}^*$, π^*_∞ and $\pi^*_{-\infty}$.

The assertions made above on the basis of geometric arguments can of course be checked and made more precise by calculation, but to save space and tedium we shall only outline selected cases. Thus to *check the existence of real roots* $\bar{h}^+$ and $\bar{h}^-$ in the cases mentioned we first write $\bar{F}=\bar{F}(h,\theta)$ in the alternative forms

$$\bar{F}=b(h-1)^2+(2/\sigma^2)(h-1)(\theta-\bar{Q}+b\sigma^2)+(2/\sigma^2)(\theta-\bar{\theta}_1), \tag{3.11}$$

$$\bar{F}=b(h-1/b)^2+(2/\sigma^2)(h-1/b)(\theta-\bar{Q}+\sigma^2)+(2/b\sigma^2)(\theta-\bar{\theta}_{1/b}). \tag{3.12}$$

If $\bar{\theta}_1>0$, we solve $\bar{F}(h,0)=0$ in the first form to obtain

$$b\sigma^2(\bar{h}^\pm-1)=\bar{Q}-b\sigma^2\pm[(\bar{Q}-b\sigma^2)^2+2b\sigma^2\bar{\theta}_1]^{\frac{1}{2}}, \tag{3.13}$$

and $\bar{\theta}_1>0$ implies the existence of distinct real roots $\bar{h}^+>1>\bar{h}^-$. If $\bar{\theta}_{1/b}>0$ we solve $\bar{F}(h,0)=0$ in the second form to obtain

$$b\sigma^2(\bar{h}^\pm-1/b)=\bar{Q}-\sigma^2\pm[(\bar{Q}-\sigma^2)^2+2\sigma^2\bar{\theta}_{1/b}]^{\frac{1}{2}}, \tag{3.14}$$

and $\bar{\theta}_{1/b}>0$ implies the existence of distinct real roots $\bar{h}^+>1/b>\bar{h}^-$.

Turning to the *characterisation of the stationary points*, we write the Jacobian matrix of $\bar{S}$ at an arbitrary point (h,θ), using obvious notation for derivatives, as

$$\begin{bmatrix} \bar{F}_h & \bar{F}_\theta \\ G_h & G_\theta \end{bmatrix} = \begin{bmatrix} (2/\sigma^2)\left(b\sigma^2 h+\theta-\bar{Q}\right) & (2/\sigma^2)\,h \\ \theta & h-1 \end{bmatrix} \tag{3.15}$$

so that the eigenvalues at a stationary point $(\bar{h},\bar{\theta})$ are given by

$$2\lambda_\pm=2\lambda_\pm(\bar{h},\bar{\theta})=\bar{F}_h+G_\theta\pm[(\bar{F}_h-G_\theta)^2+4\bar{F}_\theta G_h]^{\frac{1}{2}}.^7 \tag{3.16}$$

[7]We distinguish between the points h^+ and h^- by superscripts, but between eigenvalues at a given

For brevity we give details only for those points which define saddles. *Consider first the point* $(1, \bar{\theta}_1)$ *in case* $\bar{\theta}_1 > 0$. Here, using (4a) and (6), we have

$$\bar{F}_h = b + 2\bar{m}/b\sigma^2 = b + \bar{R}, \quad \bar{F}_\theta = 2/\sigma^2, \quad G_h = \bar{\theta}_1, \quad G_\theta = 0, \tag{3.17}$$

so that

$$2\lambda_\pm(1, \bar{\theta}_1) = b + \bar{R} \pm [(b + \bar{R})^2 + 8\bar{\theta}_1/\sigma^2]^{\frac{1}{2}}, \quad \bar{R} = 2\bar{m}/b\sigma^2. \tag{3.18}$$

Since $\bar{\theta}_1 > 0$, the eigenvalues are real and of opposing sign, with $\lambda_+ > 0 > \lambda_-$, confirming that the point is a saddle.

A calculation of the directions of 'arrival' and 'departure' of 'stable' ($z \uparrow$) and 'unstable' ($z \downarrow$) paths at the saddle point $\bar{\pi}^* = (1, \bar{\theta}_1)$, i.e. the limits of $[h(z) - 1]/[\theta(z) - \bar{\theta}_1]$ as $z \to \infty$ and $z \to -\infty$, shows that the stable limit is $\lambda_-/\bar{\theta}_1 < 0$, while the unstable limit is $\lambda_+/\bar{\theta}_1 > 0$.[8] The stable manifold $\bar{\mathcal{M}}^{\triangleright} = \bar{\mathcal{M}}^{\triangleright}(1, \bar{\theta}_1)$ may therefore be represented locally as the graph of a $\mathbf{C}^1$ function $h = \bar{f}(\theta; 1, \bar{\theta}_1)$, or simply $\bar{f}(\theta)$, with negative slope, defined on a small θ-neighbourhood of $\bar{\theta}_1$, satisfying

$$1 = \bar{f}(\bar{\theta}_1), \quad \bar{f}'(\bar{\theta}_1) = \lambda_-/\bar{\theta}_1 < 0. \tag{3.19a}$$

Similarly, the unstable manifold $\bar{\mathcal{M}}^{\triangleleft} = \bar{\mathcal{M}}^{\triangleleft}(1, \bar{\theta}_1)$ may be represented locally as the

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point by subscripts. Note that $\lambda_+(\bar{h}, \bar{\theta})$ and $\lambda_-(\bar{h}, \bar{\theta})$ denote the numbers obtained by taking respectively the positive and negative square roots in (3.16), but these need not in all cases be respectively the positive and negative eigenvalues at $(\bar{h}, \bar{\theta})$.

[8] Quick calculation (avoiding transformation of $\bar{S}$ to real canonical form, cf. Palis and de Melo [1982] Theorem 2.5). For $z \to \infty$, let

$$r = \lim[h(z) - 1]/[\theta(z) - \bar{\theta}_1]. \tag{i}$$

Then also

$$\begin{aligned} r &= \lim[h'(z)/\theta'(z)] \\ &= \lim\{[\bar{F}_h(h-1) + \bar{F}_\theta(\theta - \bar{\theta}_1)]/[G_h(h-1) + G_\theta(\theta - \bar{\theta}_1)]\} \end{aligned} \tag{ii}$$

on linearising, where $h = h(z)$, $\theta = \theta(z)$ and the partial derivatives $\bar{F}_h$, $\bar{F}_\theta$, G_h, G_θ are evaluated at $(1, \bar{\theta}_1)$. Values of these derivatives appear at (3.17). Using these and (i), (ii) reduces to

$$\begin{aligned} &r = \lim\{(1/\bar{\theta}_1)[\bar{F}_h + \bar{F}_\theta/r]\}, \text{ or simply} \\ &r^2 - r\bar{F}_h - \bar{F}_\theta = 0. \end{aligned}$$

Solving for r, taking the negative root (to obtain the 'inward' limit) and using (3.17–18), one obtains $r = \lambda_-(1, \bar{\theta}_1)/\bar{\theta}_1$. On the other hand, $r = \bar{f}'(\bar{\theta}_1)$, yielding the second expression in (3.19a). The second expression in (3.19b) is obtained similarly if the positive root is chosen.‖

graph of a $\mathbf{C}^1$ function $h = \bar{g}(\theta; 1, \bar{\theta}_1) = \bar{g}(\theta)$ with positive slope, defined on a small θ-neighbourhood of $\bar{\theta}_1$, satisfying

$$1 = \bar{g}(\bar{\theta}_1), \quad \bar{g}'(\bar{\theta}_1) = \lambda_+/\bar{\theta}_1 > 0. \tag{3.19b}$$

In particular, for $\bar{F} = F_\infty$, we have $\bar{\theta}_1 = n > 0$ — see (7), which was proposed as the limiting value of the consumption ratio $\theta(z)$ as $z \to \infty$ for Type 1 b.v.p.s. Then the saddle point is $\pi^*_\infty = (1, n)$, $\bar{\mathcal{M}}^\triangleright$ and $\bar{\mathcal{M}}^\triangleleft$ are written $\mathcal{M}^\triangleright_\infty(1, n)$ and $\mathcal{M}^\triangleleft_\infty(1, n)$, (or simply $\mathcal{M}^\triangleright_\infty$ and $\mathcal{M}^\triangleleft_\infty$ when the meaning is clear). The functions $\bar{f}$, $\bar{g}$ are written $f_\infty(\theta; 1, n)$ and $g_\infty(\theta; 1, n)$ or simply f_∞, g_∞. The eigenvalues $\lambda_\pm(1, n)$ at π^*_∞ are obtained from (18) with $\bar{m} = m$, see (7), as

$$\lambda_\pm(\infty) \cdot \sigma^2 = \frac{1}{2}b\sigma^2 + m/b \pm [(\frac{1}{2}b\sigma^2 + m/b)^2 + 2n\sigma^2]^{1/2}. \tag{3.19c}$$

Similarly, for $\bar{F} = F_{-\infty}$, we have $\bar{\theta}_1 = N > 0$ — see (8), which was proposed as the limiting value of $\theta(z)$ as $z \downarrow -\infty$ for Type 1 b.v.p.s. In this case, the saddle point is $\pi^*_{-\infty} = (1, N)$, $\bar{\mathcal{M}}^\triangleright$ and $\bar{\mathcal{M}}^\triangleleft$ are written $\mathcal{M}^\triangleright_{-\infty}(1, N)$ and $\mathcal{M}^\triangleleft_{-\infty}(1, N)$ or simply $\mathcal{M}^\triangleright_{-\infty}$ and $\mathcal{M}^\triangleleft_{-\infty}$. Also $\bar{f}$, $\bar{g}$ are written $f_{-\infty}(\theta; 1, N)$, $g_{-\infty}(\theta; 1, N)$ or simply $f_{-\infty}$, $g_{-\infty}$, and the eigenvalues $\lambda_\pm(1, N)$ at $\pi^*_{-\infty}$ are calculated from (18) with $\bar{m} = m - \psi'_0$, see (8). This yields

$$\lambda_\pm(-\infty) \cdot \sigma^2 = \frac{1}{2}b\sigma^2 + (m - \psi'_0)/b \pm [(\frac{1}{2}b\sigma^2 + (m - \psi'_0)/b)^2 + 2N\sigma^2]^{1/2}. \tag{3.19d}$$

Turning to stationary points $(\bar{h}, \bar{\theta})$ *of* $(\bar{F}, G)$ *with* $\bar{\theta} = 0$, we have, using (3) and (10),

$$\begin{aligned} \bar{F}_h &= (2/\sigma^2)(b\sigma^2\bar{h} - \bar{Q}) = \pm(2/\sigma^2)[\bar{Q}^2 + 2\bar{m}\sigma^2]^{\frac{1}{2}}, \\ \bar{F}_\theta &= (2/\sigma^2)\bar{h}, \quad G_h = 0, \quad G_\theta = \bar{h} - 1, \end{aligned} \tag{3.20}$$

where $\bar{h}$ is $\bar{h}^+$ or $\bar{h}^-$ and the sign of the square root in the expression for $\bar{F}_h$ is chosen as $+$ or $-$ according to whether $\bar{h}$ is $\bar{h}^+$ or $\bar{h}^-$.[9] Now (16) yields

$$\lambda_+(\bar{h}, 0) = \bar{F}_h, \quad \lambda_-(\bar{h}, 0) = G_\theta. \tag{3.21}$$

[9] The root must be real because $\bar{h}^\pm$ are real, but it can also be checked directly that $\bar{Q}^2 + 2\bar{m}\sigma^2 > 0$ in case either $\bar{\theta}_1 > 0$ or $\bar{\theta}_{1/b} > 0$. If $\bar{m} > 0$, this is obvious. If $\bar{m} \leq 0$, express $\bar{Q}$ in terms of $\bar{\theta}_1$ or $\bar{\theta}_{1/b}$ using (3.4–5) and rearrange to represent $\bar{Q}^2 + 2\bar{m}\sigma^2$ as the sum of perfect squares and a positive term. Explicitly, if $\bar{m} \leq 0$ and $\bar{\theta}_1 > 0$, 3.4 yields

$$\bar{Q}^2 + 2\bar{m}\sigma^2 = (\bar{\theta}_1 + \frac{1}{2}b\sigma^2 - \bar{m}/b)^2 + 2\bar{m}\sigma^2 = \bar{\theta}_1^2 + 2\bar{\theta}(\frac{1}{2}b\sigma^2 - \bar{m}/b) + (\frac{1}{2}b\sigma^2 + \bar{m}/b)^2.$$

If $b > 1$ and $\bar\theta_{1/b} > 0 > \bar\theta_1$ selecting $\bar h = \bar h^+$ yields $\lambda_+ = \bar F_h > 0$, $\lambda_- = \bar h^+ - 1 < 0$ by Prop. 4, confirming the saddle property of $[\bar h^+, 0)$. In this case the unstable curve $\bar{\mathcal{M}}^{\triangleleft}(\bar h^+, 0)$ is on the vertical axis and there is no representing function of the form $h = \bar g(\theta)$.

For a solution $(h(z), \theta(z))$ with $\theta(z) > 0$, converging to $(\bar h^+, 0)$, the limit of $[h(z) - \bar h^+]/\theta(z)$ as $z \uparrow \infty$ may be calculated as $2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) < 0$, where $\lambda_\pm = \lambda_\pm(\bar h^+, 0)$, so that the stable manifold $\bar{\mathcal{M}}^{\triangleright}(\bar h^+, 0)$ may be represented locally as the graph of a $\mathbf{C}^1$ function $h = \bar f(\theta; \bar h^+, 0) = \bar f(\theta)$ with negative slope defined on a (left closed) right neighbourhood of $\bar\theta = 0$ satisfying[10]

$$\bar h^+ = \bar f(0), \quad \bar f'(0) = 2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) < 0. \tag{3.22a}$$

In particular, if $\bar F = F_\infty$, we have

$$\bar\theta_{1/b} = q > 0, \quad \bar\theta_1 = n < 0, \quad \bar Q = Q, \quad \bar m = m, \text{ see (7).}$$

Then the saddle point is $\pi^*_\infty = (h^+_\infty, 0)$, $\bar{\mathcal{M}}^{\triangleright} = \mathcal{M}^{\triangleright}_\infty(h^+_\infty, 0) = \mathcal{M}^{\triangleright}_\infty$, $\bar f = f_\infty(\theta; \bar h^+, 0) = f_\infty$, and the values of $\bar h^+ = h^+_\infty$ and of $\lambda_\pm$ are obtained from (10) and (20–21). Thus

$$\begin{aligned} \lambda_+(h^+_\infty, 0) &= (2/\sigma^2)(b\sigma^2 h^+_\infty - Q) = +(Q^2 + 2m\sigma^2)^{1/2} > 0, \\ \lambda_-(h^+_\infty, 0) &= (h^+_\infty - 1) < 0. \end{aligned} \tag{3.22b}$$

Since $1/b < h^+_\infty < 1$ by Prop.4, the point $(h^+_\infty, 0)$ satisfies the conditions (0.13) for the limit of the optimal consumption function (and this remains true if $h^+_\infty = 1$). Now (22a)

If $\bar m \le 0$ and $\bar\theta_{1/b} \ge 0$, (3.5) yields

$$\bar Q^2 + 2\bar m\sigma^2 = (\bar\theta_{1/b} + \frac{1}{2}\sigma^2 - \bar m)^2 + 2\bar m\sigma^2 = \bar\theta^2_{1/b} + 2\bar\theta_{1/b}[\frac{1}{2}\sigma^2 - \bar m] + (\frac{1}{2}\sigma^2 + \bar m)^2.$$

[10] Quick calculation, proceeding as in fn.8. For $z \to \infty$, let

$$\begin{aligned} r &= \lim(h(z) - \bar h^+)/\theta(z). \text{ Then also} \\ r &= \lim[h'(z)/\theta'(z)] \\ &= \lim\{[\bar F_h(h - \bar h^+) + \bar F_\theta \cdot \theta]/G_\theta \cdot \theta\} \\ &= [\lambda_+ r + (2/\sigma^2)\bar h^+]/[\bar h^+ - 1]. \end{aligned}$$

Rearranging, using $\lambda_- = \bar h^+ - 1$ and noting that $r = \bar f'(0)$, this yields (3.22a). The corresponding calculation for (3.23a) below is similar, with $z \to \infty$ replaced by $z \to -\infty$, f by g, but of course the eigenvalues are different.

Results corresponding to (3.22c) and (3.23c) were given incorrectly in Foldes [1996] at eqns.(3.22) and (3.23).

appears as

$$\text{(3.22c)}\quad h^+_\infty = f_\infty(0), \quad f'_\infty(0) = 2(1+\lambda_-)/\sigma^2(\lambda_- - \lambda_+) < 0, \text{ where } \lambda_\pm = \lambda_\pm(h^+_\infty, 0).$$

Similarly, if $b < 1$ and $\bar\theta_{1/b} > 0 > \bar\theta_1$, selecting $\bar h = \bar h^-$ in (21) yields $\lambda_+ = \bar F_h < 0$, $\lambda_- = \bar h^- - 1 > 0$ by Prop.4, a saddle at $(\bar h^-, 0)$. This time the stable curve $\bar{\mathcal{M}}^{\triangleright}(\bar h^-, 0)$ is on the vertical axis and there is no representing function $h = \bar f(\theta)$. For a solution $(h(z), \theta(z))$, with $\theta(z) > 0$, converging to $(\bar h^-, 0)$ the limit of $(h(z) - \bar h^-)/\theta(z)$ as $z \to -\infty$ is calculated as $2(1+\lambda_-)/\sigma^2(\lambda_- - \lambda_+) > 0$, where now $\lambda_\pm = \lambda_\pm(\bar h^-, 0)$, so that the unstable manifold (w.r.t. the forward motion) $\bar{\mathcal{M}}^{\triangleleft}(\bar h^-, 0)$ may be represented locally as the graph of a $\mathbf{C}^1$ function $h = \bar g(\theta; \bar h^-, 0) = \bar g(\theta)$ with positive slope, defined on a (left closed) right neighbourhood of $\bar\theta = 0$, satisfying

$$\text{(3.23a)}\qquad \bar h^- = \bar g(0), \quad \bar g'(0) = 2(1+\lambda_-)/\sigma^2(\lambda_- - \lambda_+).$$

In particular, if $\bar F = F_{-\infty}$, we have

$$\bar\theta_{1/b} = q > 0, \quad \bar\theta_1 = N < 0, \quad \bar Q = Q + \psi'_0, \quad \bar m = m - \psi'_0, \quad \text{see (8).}$$

Then the saddle point is $\pi^*_{-\infty} = (h^-_{-\infty}, 0)$, $\bar{\mathcal{M}}^{\triangleleft} = \mathcal{M}^{\triangleleft}_{-\infty}(h^-_{-\infty}, 0) = \mathcal{M}^{\triangleleft}_{-\infty}$, $\bar g = g_{-\infty}(\theta; h^-_{-\infty}, 0) = g_{-\infty}$, and the values of $\bar h_-$ and $\lambda_\pm$ are obtained from (10) and (20–21). Thus

(3.23b)

$$\lambda_+(h^-_{-\infty}, 0) = (2/\sigma^2)(b\sigma^2 h^-_{-\infty} - Q - \psi'_0) = -[(Q+\psi'_0)^2 + 2(m-\psi'_0)\sigma^2]^{1/2} < 0,$$
$$\lambda_-(h^-_{-\infty}, 0) = h^-_{-\infty} - 1 > 0.$$

Since $1/b > h^-_{-\infty} > 1$ by Prop.4, the point $(h^-_{-\infty}, 0)$ satisfies the condition (0.14) for the limit of an optimal consumption function, (also if $h^-_{-\infty} = 1$). Now (23a) appears as

$$\text{(3.23c)}\quad h^-_{-\infty} = g_{-\infty}(0), \quad g'_{-\infty} = 2(1+\lambda_-)/\sigma^2(\lambda_- - \lambda_+) > 0, \text{ where } \lambda_\pm = \lambda_\pm(h^-_{-\infty}, 0).$$

If $\bar\theta_1 = 0$, with either $b > 1$, $\bar h^+ = 1$, or $b < 1$, $\bar h^- = 1$, the point (1,0) is a saddle-node but the preceding remarks apply with routine changes; see fn. 6.

This discussion yields the following important result:

PROPOSITION 5. The points satisfying the boundary conditions at $z = \infty$ and $z = -\infty$ prescribed by Theorem 4A are precisely the saddle points of S_∞ and $S_{-\infty}$. In particular, the condition (0.13) cannot be satisfied by any $h^*(+\infty)$ other than h^+_∞, and (0.14) cannot be satisfied by any $h^*(-\infty)$ other than $h^-_{-\infty}$.

PROOF. A star solution must by definition be a solution of S defined on the whole of $\Re$ and converging to finite limits as $z \to \pm\infty$. According to Prop. 2(ii) the limits must be stationary points of S_∞ and $S_{-\infty}$. For Type 1 b.v.p.s the result is immediate because (0.11) and (0.12) give the precise co-ordinates of the relevant saddle points. In the case of Type 0 b.v.p.s the saddle points $(h^+_\infty, 0)$ and $(h^-_{-\infty}, 0)$ satisfy (0.13) and (0.14) respectively, and Prop. 4 shows that these are the only stationary points of S_∞ and $S_{-\infty}$ in the prescribed intervals. (Once again, 'saddle' here includes 'saddle-node'.)‖
An appropriate restatement of Theorem 4 is given at the beginning of Section 4.

For convenience of reference, the data for saddle points π^*_∞ of S_∞ and $\pi^*_{-\infty}$, for both Types, are summarised in the following Table I. The matrix in each Section of the Table is the Jacobian matrix (3.15) evaluated at π^*_∞ or $\pi^*_{-\infty}$. This is followed by expressions for the eigenvalues at the point in question and for the slope of f_∞ at π^*_∞ and of $g_{-\infty}$ at $\pi^*_{-\infty}$.

$$\begin{bmatrix} (2/\sigma^2)(m/b + \frac{1}{2}b\sigma^2) & (2/\sigma^2) \\ n & 0 \end{bmatrix}$$

$$\lambda_\pm \sigma^2 = \frac{1}{2}b\sigma^2 + m/b \pm [(\frac{1}{2}b\sigma^2 + m/b)^2 + 2n\sigma^2]^{\frac{1}{2}}$$

Note $\lambda_+ \cdot \lambda_- = -2n\sigma^2$

$$f'_\infty(n) = \lambda_-/n < 0$$

<u>Data for Saddle Point of S_∞ at</u> $(1, n)$, $n > 0$ (Type 1)

$$\begin{bmatrix} (2/\sigma^2)[(m - \psi'_0)/b + \frac{1}{2}b\sigma^2] & (2/\sigma^2) \\ N & 0 \end{bmatrix}$$

$$\lambda_\pm \sigma^2 = \frac{1}{2}b\sigma^2 + [(m - \psi'_0)/b] \pm [\{\frac{1}{2}b\sigma^2 + (m - \psi'_0)/b\}^2 + 2N\sigma^2]^{\frac{1}{2}}$$

Note $\lambda_+ \cdot \lambda_- = -2N\sigma^2$

$$g'_{-\infty}(N) = \lambda_+/N < 0$$

<u>Data for Saddle Point of $S_{-\infty}$ at</u> $(1, N)$, $N > 0$ (Type 1)

$$\begin{bmatrix} (2/\sigma^2)(b\sigma^2 h^+_\infty - Q) & (2/\sigma^2)h^+_\infty \\ 0 & h^+_\infty - 1 \end{bmatrix}$$

$\lambda_+ = (2/\sigma^2)(b\sigma^2 h^+_\infty - Q) > 0;$

$b\sigma^2 h^+_\infty = Q + [Q^2 + 2m\sigma^2]^{1/2}$ (positive square root)

$\lambda_- = h^+_\infty - 1 < 0.$ Note $h^+_\infty > 1/b$

$$f'_\infty(0) = 2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) < 0$$

<u>Data for Saddle Point of S_∞ at</u> $(h^+_\infty, 0)$, $q > 0 > n$, $b > 1$ (Type 0)

$$\begin{bmatrix} (2/\sigma^2)(b\sigma^2 h^-_{-\infty} - Q - \psi'_0) & (2/\sigma^2)h^-_{-\infty} \\ 0 & h^-_{-\infty} - 1 \end{bmatrix}$$

$\lambda_+ = (2/\sigma^2)(b\sigma^2 h^-_{-\infty} - Q - \psi'_0);$

$b\sigma^2 h^-_{-\infty} = Q + \psi'_0 - [(Q + \psi'_0)^2 + 2(m - \psi'_0)\sigma^2]^{1/2}$ (positive square root)

$\lambda_- = h^-_{-\infty} - 1 > 0.$ Note $h^-_{-\infty} < 1/b.$

$$g'_{-\infty}(0) = 2(1 + \lambda_-)/\sigma^2(\lambda_- - \lambda_+) > 0$$

<u>Data for Saddle Point of $S_{-\infty}$ at</u> $(h^-_{-\infty}, 0)$, $q > 0 > N$, $b < 1$ (Type 0)

TABLE 1: DATA FOR SADDLE POINTS OF S_∞ AND $S_{-\infty}$

(iii) *Dynamics and Asymptotic Behaviour of Solutions of Systems* $\bar{S} = (\bar{F}, G)$.

We next note some properties of the phase behaviour of solutions of $\bar{S}$, in particular their limiting behaviour. Where the assertions are elementary or sufficiently obvious from the Figures we omit formal proofs.

Consider again the system $\bar{S}$: $h' = \bar{F}(h, \theta)$, $\theta' = (h-1)\theta$, see (3 a,b), and take as domain $\{\theta > 0\}$. For a given solution of $\bar{S}$, let (z_-, z_+) denote its maximal interval of definition. We consider solutions of $\bar{S}$, and their paths, restricted to intervals of the form $[z_\diamondsuit, z_+)$ or $(z_-, z_\diamondsuit]$ for given $z_\diamondsuit \in \Re$, and show that *every path converges to a limit, finite or infinite, as* $z \uparrow$ *or* $z \downarrow$. For this purpose we review the directions of motion within and between various phase regions. (By a phase region we mean an open connected set of $\{\theta > 0\}$ on which $\bar{F}$ and G both have constant signs, the boundary consisting of arcs of $\bar{F} = 0$ or $G = 0$ or both.) The regions are shown in Figures 2, each with its pair of phase arrows. Cases with $\bar{m} = 0$, and those with $b = 1$ or $\bar{\theta}_1 = 0$ are not depicted but unless stated offer no significant exception to what follows. The only path never entering any phase region is the stationary path $(1, \bar{\theta}_1)$ if $\bar{\theta}_1 > 0$; this may be left aside. According to earlier discussion, no path can terminate in the interior of a phase region. Within each phase region a path is monotone in both co-ordinates; thus it is enough to check that each path is ultimately in one of the regions as $z \uparrow z_+$ or $z \downarrow z_-$. In fact, a review of phase transitions yields more: (a) A path which passes through a point on the boundary between two phase regions immediately enters one of them. (b) There is a one-way flow between regions as $z \uparrow$, also as $z \downarrow$. (c) A path which once leaves a region, as $z \uparrow$ or as $z \downarrow$, cannot return to it via a sequence of other regions. The existence of limits for paths follows.[11]

As regards *the values of the limits*, the possibilities in the case of *bounded solutions* are few. (Now we usually leave aside stationary solutions and those with paths in $\{\theta = 0\}$.)

[11]For completeness, consider solutions with paths in the vertical axis $\{\theta = 0\}$. These solutions satisfy the equation $h' = \bar{F}(h, 0)$. According to Prop.4, there are two points $\bar{h}^\pm$ defining stationary solutions, i.e. satisfying $0 = \bar{F}(h, 0)$. These points partition the rest of the axis into three intervals, denoted $\bar{I}^+ = (\bar{h}^+, \infty)$, $\bar{I}^0 = (\bar{h}^-, \bar{h}^+)$ and $\bar{I}^- = (-\infty, \bar{h}^-)$. Clearly $\bar{F}(h, 0)$ is positive on $\bar{I}^+$ and on $\bar{I}^-$, negative on $\bar{I}^0$, and a non-stationary solution can take values in only one of these intervals. More precisely, for $z \uparrow z_+$, solutions with paths $h(z)$ in $\bar{I}^-$ increase, those in $\bar{I}^0$ decrease, and in both cases $z_+ = \infty$ and the limit is $\bar{h}^-$; for $h(z)$ in $\bar{I}^+$, $h(z)$ increases without bound to $h(z_+) = \infty$. For $z \downarrow z_-$, paths in $\bar{I}^+$ decrease, those in $\bar{I}^0$ increase, with $z_- = -\infty$ and limit $\bar{h}^+$ in both cases, while paths in $\bar{I}^-$ decrease without bound to $h(z_-) = -\infty$.

In particular, if $\bar{F} = F_\infty$, the points h^+_∞ and h^-_∞ define stationary solutions, while all non-stationary solutions which remain bounded as $z \uparrow z_+$ ($z \downarrow z_-$) converge to h^-_∞ (h^+_∞). If $\bar{F} = F_{-\infty}$, the points $h^+_{-\infty}$ and $h^-_{-\infty}$ define stationary solutions, while all non-stationary solutions which are bounded for $z \uparrow z_+$ ($z \downarrow z_-$) converge to $h^-_{-\infty}$ ($h^+_{-\infty}$).

If a solution is bounded as $z \uparrow$ ($z \downarrow$) then $z_+ = \infty$ ($z_- = -\infty$) — see Prop. 1. The limit must be a stationary point, say $(\bar{h}, \bar{\theta})$, and for a forward (backward) limit it must satisfy $\bar{h} \le 1$ ($\bar{h} \ge 1$) since otherwise the arrows point the wrong way. Thus in Type 1 systems the only possible finite forward limits are $(1, \bar{\theta}_1)$ and $(\bar{h}^-, 0)$, the only backward ones are $(1, \bar{\theta}_1)$ and $(\bar{h}^+, 0)$ — see Figs. 2(i) and (ii). There are no finite backward limits in Type 0 systems with $b > 1$, but there are forward limits $(\bar{h}^\pm, 0)$ — see Figs. 2(iii) and (iv). Similarly there are no finite forward limits in Type 0 systems with $b < 1$, but there are backward limits $(\bar{h}^\pm, 0)$ — see Fig. 2(v). In particular, these results apply to $\bar{F} = F_\infty$ with $\bar{\theta}_1 = n$, $\bar{h}^\pm = h^\pm_\infty$, and to $\bar{F} = F_{-\infty}$ with $\bar{\theta}_1 = N$, $\bar{h}^\pm = h^\pm_{-\infty}$.

As previously mentioned, the Figures are also classified according to the sign of $\bar{m}$. If $\bar{m} > 0$, forward motion on the axis $\{h = 0\}$ is always downward, if $\bar{m} < 0$ it is upward, and if $\bar{m} = 0$ the motion is along the axis, which acts as a barrier. Also, in cases with $\bar{m} > 0$, $\bar{h}^+$ and $\bar{h}^-$ have opposite signs, whereas with $\bar{m} < 0$ they have the same sign; in the latter case all finite limits of paths, forward or backward, have both co-ordinates positive, (non-negative if $\bar{m} \le 0$).

To see some useful consequences of these remarks, consider the 'invariant' paths at the saddle $\bar{\pi}^* = (1, \bar{\theta}_1)$ in Figs. 2(i) and (ii), representing Type 1 systems. It has been noted above that the stable manifold $\bar{\mathcal{M}}^\triangleright$ can be represented locally by a $\mathbf{C}^1$ function $\bar{h} = \bar{f}(\theta)$ with negative slope. This manifold consists of two paths (in addition to the saddle point). Continuing the left path backward as $z \downarrow$, it is seen that $\theta \downarrow$ and $h \uparrow$ with no possible phase transitions, so the backward limit must be $(\bar{h}^+, 0)$ with $z_- = -\infty$. Tracing the right path backward, one has $\theta \uparrow$, $h \downarrow$ as long as h stays positive, which it must do forever if $\bar{m} \ge 0$ as in Fig. 2(ii); then the path limit is given by $h \downarrow 0$, $\theta \uparrow \infty$, and again $z_- = -\infty$ since h stays bounded. If $\bar{m} < 0$ as in Fig. 2(i), the path cannot stay in the domain $\{h > 0\}$ forever but crosses at some $\theta = \bar{\theta}_+$ into $\{h < 0\}$, where it remains and continues moving to the right. Thereafter the path eventually crosses the curve $\bar{F}^-(0)$ (i.e. the negative branch of the hyperbola $\bar{F}(h, \theta) = 0$), after which the path limit is $h \uparrow 0$, $\theta \uparrow \infty$, again with $z_- = -\infty$. In all these cases the representation $h = \bar{f}(\theta)$, with $\bar{f}$ of class $\mathbf{C}^1$, can be continued for all $\theta \in (0, \infty)$, (global stable manifold). We have $\bar{f}(\theta) > 0$ for all θ if $\bar{m} \ge 0$, but $\bar{f}(\theta) < 0$ for θ greater than some $\bar{\theta}_+ < \infty$ if $\bar{m} < 0$; if $\bar{f}(\theta) > 0$ for all $\theta > 0$, we set $\bar{\theta}_+ = \infty$. A further useful remark is that the curve

$$\bar{\mathcal{M}}^\triangleright = \{h = \bar{f}(\theta)\} \tag{3.24a}$$

separates two open 'half-spaces' within $\{\theta > 0\}$, say

$$\bar{\mathcal{U}}^{\triangleright} = \{h > \bar{f}(\theta)\}, \quad \bar{\mathcal{B}}^{\triangleright} = \{h < \bar{f}(\theta)\}. \tag{3.24b}$$

Paths of $\bar{S}$ which ever enter the lower half-space $\bar{\mathcal{B}}^{\triangleright}$ as $z \uparrow$ are bounded for the forward motion and converge to $(\bar{h}^-, 0)$ as $z \uparrow$, with $z_+ = \infty$, while paths which ever enter $\bar{\mathcal{U}}^{\triangleright}$ pass ultimately into the region $\{\bar{F} > 0, G > 0\} = \{\bar{F} > 0, h > 1\}$ and so become unbounded (with $z_+ < \infty$, see below).

Consider now the unstable manifold $\bar{\mathcal{M}}^{\triangleleft}$, which is represented locally by a $\mathbf{C}^1$ function $h = \bar{g}(\theta)$ with positive slope. This manifold again consists of two paths (in addition to the saddle point). If the left path is continued forward as $z \uparrow$, it is seen that $\theta \downarrow$ and $h \downarrow$ as long as the path remains in $\{h \geq 0\}$, which goes on for all $\theta > 0$ if $\bar{m} \leq 0$ as in Fig.2(i); but if $\bar{m} > 0$ as in Fig.2(ii) the path must cross into $\{h < 0\}$ at some $\theta = \bar{\theta}_- > 0$ and then eventually pass into a region with $\bar{F} > 0$, after which h increases again while remaining negative. In either case the path limit is $(\bar{h}^-, 0)$ and $z_+ = \infty$. As to the right path, this passes immediately into the region $\{\bar{F} > 0, G > 0\}$ and eventually becomes unbounded with $h \uparrow \infty$, $\theta \uparrow \infty$ as $z \uparrow z_+$ (with $z_+ < \infty$ see below for details). The representation $h = \bar{g}(\theta)$ can be continued for all $\theta > 0$ (global unstable manifold), with $\bar{g}(\theta) > 0$ for all θ if $\bar{m} \leq 0$, but $\bar{g}(\theta) < 0$ for θ less than some $\bar{\theta}_-$ if $\bar{m} > 0$; if $\bar{g}(\theta) > 0$ for all $\theta > 0$, we set $\bar{\theta}_- = 0$. In particular, *whatever* $\bar{m}$, *one of the curves* $\bar{f}$ *and* $\bar{g}$ *always stays positive for* $\theta \in (0, \infty)$, (but both have this property only if $\bar{m} = 0$). The curve

$$\bar{\mathcal{M}}^{\triangleleft} = \{h = \bar{g}(\theta)\} \tag{3.25a}$$

separates two open half-spaces

$$\bar{\mathcal{U}}^{\triangleleft} = \{h < \bar{g}(\theta)\}, \quad \bar{\mathcal{B}}^{\triangleleft} = \{h > \bar{g}(\theta)\} \tag{3.25b}$$

in $\{\theta > 0\}$, with paths entering $\bar{\mathcal{B}}^{\triangleleft}$ bounded for the backward motion and converging to $(\bar{h}^+, 0)$ as $z \downarrow -\infty$, while for paths entering $\bar{\mathcal{U}}^{\triangleleft}$ we have $h \to 0$, $\theta \to \infty$ as $z \downarrow z_- = -\infty$ (see below for details).

A similar analysis can be carried out for Type 0 systems. In cases with $b > 1$ and $\bar{\theta}_{1/b} > 0 > \bar{\theta}_1$, as in Figs.2(iii–iv), the stable manifold at $\bar{\pi}^* = (\bar{h}^+, 0)$ has only one branch lying in $\{\theta \geq 0\}$, which we again denote by $\bar{\mathcal{M}}^{\triangleright}$. Its representation by $h = \bar{f}(\theta)$ can be continued for all $\theta \geq 0$, with a negative slope while h remains positive. This goes on for all $\theta \geq 0$ if $\bar{m} \geq 0$ as in Fig. 2(iv) (and we set $\bar{\theta}_+ = \infty$); but if $m < 0$ as in

Fig. 2(iii) then h becomes negative for θ greater than some finite $\bar\theta_+$, with $h \uparrow 0$, $\theta \uparrow \infty$ as $z \uparrow z_+ = \infty$ thereafter as described above. Once again, all paths of $\bar S$ which ever enter $\bar{\mathcal{B}}^\triangleright$ go to $(\bar h^-, 0)$ as $z \uparrow z_+ = \infty$ while paths entering $\bar{\mathcal{U}}^\triangleright$ become unbounded upward. *The unstable manifold is on the vertical axis so that the function $\bar g$ is undefined, and we set $\bar{\mathcal{U}}^\triangleleft = \{\theta > 0\}$.*

In case $b < 1$ and $\bar\theta_{1/b} > 0 > \bar\theta_1$ as in Fig. 2(v), the unstable manifold at $\bar\pi^* = (\bar h^-, 0)$ has only one branch lying in $\{\theta \geq 0\}$, which is denoted by $\bar{\mathcal{M}}^\triangleleft$. Its representation by $h = \bar g(\theta)$ can be continued for all $\theta \geq 0$ (and we set $\bar\theta_- = 0$), but this time the only admissible possibility is that $\bar m < 0$; thus the curve $\bar g$ has positive slope for all θ and becomes unbounded as $z \uparrow$, with $h \uparrow \infty$, $\theta \uparrow \infty$ (as $z \uparrow z_+ < \infty$). For $z \downarrow$, the paths in $\bar{\mathcal{B}}^\triangleleft$ converge to $(\bar h^+, 0)$ and those in $\bar{\mathcal{U}}^\triangleleft$ to $(0, \infty)$, with $z_- = -\infty$. *Now the stable manifold is on the vertical axis so that the function $\bar f$ is undefined, and we set $\bar{\mathcal{U}}^\triangleright = \{\theta > 0\}$.*

It remains to give a brief account of the *asymptotic behaviour of unbounded solutions of $\bar S$* with paths in $\{\theta > 0\}$. (The details are not needed for later proofs and it is possible to skip to Prop. 6.)

Consider first a solution (h, θ) of $\bar S$ which is unbounded as $z \uparrow$. As the Figures and previous discussion show, the corresponding path is ultimately in $\bar U^\triangleright$ and we may further assume that it is ultimately in the phase region $\{\bar F > 0, G > 0\} = \{\bar F > 0, h > 1\}$, say for $z_\diamondsuit \leq z < z_+$. Specifically, choose $(h_\diamondsuit, \theta_\diamondsuit) = (h(z_\diamondsuit), \theta(z_\diamondsuit))$ in this region and set

$$\phi(z; h_\diamondsuit, \theta_\diamondsuit, z_\diamondsuit) = (h(z), \theta(z); z_\diamondsuit \leq z < z_+).$$

Now $h(z)$ and $\theta(z)$ are always increasing for $z \in [z_\diamondsuit, z_+)$, so that the monotone (finite or infinite) limits $h(z_+)$ and $\theta(z_+)$ exist. Note that

$$\begin{aligned}&(\ln\theta(z))' = \theta'(z)/\theta(z) = h(z) - 1 > h_\diamondsuit - 1 > 0 \text{ for } z \in (z_\diamondsuit, z_+), \text{ hence}\\ &\ln[\theta(z)/\theta_\diamondsuit] > (z - z_\diamondsuit)(h(z_\diamondsuit) - 1) \uparrow (z_+ - z_\diamondsuit)(h_\diamondsuit - 1).\end{aligned}$$

By assumption, at least one of $h(z)$ or $\theta(z)$ is unbounded as $z \uparrow$. If $h(z)$ were bounded, we should have $z_+ = \infty$ by Prop.1, with $1 < h(z_+) = h(\infty) < \infty$, hence $\theta(z_+) = \theta(\infty) = \infty$; however (3.3) would then imply $h'(z) = \bar F[h(z), \theta(z)] \to \infty$ as $z \uparrow \infty$, hence $h(\infty) = \infty$, a contradiction. So $h(z_+) = \infty$ and either $z_+ = \infty$ or $z_+ < \infty$. In the former case, it is immediate that $h(z)$ and $\theta(z)$ are defined for all

$z \in [z_\diamondsuit, \infty)$ with $h(z_+) = \infty$, $\theta(z_+) = \infty$. In the latter case, we have

$$\ln[\theta(z_+)/\theta_\diamondsuit] = \int_{[z_\diamondsuit, z_+]} [h(z) - 1] dz = \lim_{z \uparrow z_+} \int_{[z_\diamondsuit, z)} [h(s) - 1] ds,$$

which must be ∞ if $h(z_+) = \infty$, cf. Natanson [1955] Ch.VI, T.1. So $\theta(z)$ is defined up to z_+ with $\theta_\diamondsuit \leq \theta(z) < \theta(z_+) = \infty$. Also, of course, $h(z)$ is defined up to z_+ with $h_\diamondsuit \leq h(z) < h(z_+) = \infty$.

It can further be shown that $z_+ < \infty$. Indeed, if $z_+ = \infty$, we could choose $z_\diamondsuit$ so large that $(h(z), \theta(z))$ is in the region $\{\bar{F} > 0, G > 0\}$ for all $z \geq z_\diamondsuit$ and also, by (3.3), that $\bar{F}[h(z), \theta(z)] > bh^2(z)$. Thus the solution of $h' = F(h, \theta)$ with $h(z_\diamondsuit) = h_\diamondsuit$, $\theta(z_\diamondsuit) = \theta_\diamondsuit$ would exceed the solution of $h' = bh^2$ with $h(z_\diamondsuit) = h_\diamondsuit$ on some initial interval $(z_\diamondsuit, z_1)$. However, the solution of $h' = bh^2$ explodes to ∞ at some $z_\diamondsuit < \infty$, so that the solution of $h' = F(h, \theta)$ would also explode at or before $z_\diamondsuit$, i.e. $z_+ < z_\diamondsuit$, a contradiction.

To investigate further the asymptotic behaviour of $\phi = (h, \theta)$ as $z \uparrow z_+$, it is convenient to take $\theta \in (\theta_\diamondsuit, \infty)$ as path parameter and to introduce a new variable

$$\zeta = (h-1)/\theta, \text{ where } \zeta = \zeta(\theta), \quad h = h(\theta), \quad \theta = \theta(z),$$

which is defined and positive for $\theta \geq \theta_\diamondsuit = \theta(z_\diamondsuit)$, $h \geq h_\diamondsuit = h(z_\diamondsuit)$, and we may take $\theta_\diamondsuit$ arbitrarily large. Abridging the notation and differentiating w.r.t. θ we have $\theta \; d\zeta = dh/d\theta = \bar{F}/G - \zeta$, where $\bar{F} = F[h(\theta), \theta]$, $G = \theta' = \theta[h(\theta) - 1]$. Further, writing $\bar{F}$ as a function of $h - 1$ as in (3.11), dividing by $G = \theta' = (h-1)\theta$ and simplifying, we get

$$\theta \; d\zeta/d\theta = (b-1)\zeta + (2/\sigma^2)[1 - (\bar{Q} - b\sigma^2)/\theta + (\theta - \bar{\theta}_1)/\theta(h(\theta) - 1)]. \tag{3.26}$$

The term in square brackets is bounded, for large θ and $h(\theta)$, by $1 \pm \gamma$, where γ is a (small) constant. Now the equation

$$\theta \; d\zeta/d\theta = (b-1)\zeta + (2/\sigma^2)(1 \pm \gamma), \quad \theta > \theta_\diamondsuit > 0, \tag{3.27}$$

has the solution

$$\zeta(\theta) = C\theta^{b-1} + (2/\sigma^2)(1 \pm \gamma)/(1-b) \qquad \text{if} \quad b \neq 1, \tag{3.28}$$

$$\zeta(\theta) = C + (2/\sigma^2)(1 \pm \gamma) \ln \theta \qquad \text{if} \quad b = 1, \tag{3.29}$$

see Kamke [1943] p.311, eq. 1.94., where $C = C(\gamma)$ is a constant to be determined from the initial condition $(\zeta(\theta_\diamondsuit), \theta_\diamondsuit)$.

Now γ can be made arbitrarily small by choosing $\theta_\diamondsuit$, and hence $h_\diamondsuit$ large enough (along the given path). The asymptotic slope of the path can then be calculated informally as follows. For $b < 1$, let $\theta \to \infty$ in (28), followed by $\gamma \downarrow 0$ (corresponding to $\theta_\diamondsuit \uparrow \infty$) to obtain $\zeta(z_+) = 2/(1-b)\sigma^2$. For $b = 1$, divide both sides of (29) by $\ln\theta$, let $\theta \to \infty$ and then $\gamma \downarrow 0$ to obtain $\zeta(\theta)/\ln\theta \to 2/\sigma^2$ as $z \uparrow z_+$, so $\zeta(\theta) \to \infty$. For $b > 1$, we have $\zeta(\theta) \to \infty$ as $\theta \to \infty$. Also, dividing both sides of (28) by θ^{b-1}, noting that

$$C = [(\zeta_\diamondsuit + 2/\sigma^2)(1 \pm \gamma)/(b-1)]\theta_\diamondsuit^{1-b} > 0 \quad \text{for } \gamma \text{ small enough} \tag{3.30}$$

and letting $\theta \to \infty$ yields

$$\zeta/\theta^{b-1} = (h-1)/\theta^b \to C \sim [\zeta_\diamondsuit + 2/\sigma^2(b-1)]\theta_\diamondsuit^{1-b} = (h_\diamondsuit - 1)\theta_\diamondsuit^{-b} + [2/\sigma^2(b-1)]\theta_\diamondsuit^{1-b}. \tag{3.31}$$

Consider now *solutions which are unbounded as* $z \downarrow$. It is clear from the phase diagrams (with the arrows reversed) that in all cases the corresponding paths are ultimately in $\bar{\mathcal{U}}^\triangleleft$ and that the following types of asymptotic behaviour are possible:

(a) If $\bar{m} > 0$, as in Figs. 2(ii) and (iv), there are paths which are ultimately in $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} > 0, h > 0\}$, and then the limiting behaviour as $z \downarrow$ is $\theta \uparrow \infty$, $h \downarrow 0$, hence $z_- = -\infty$.

(b) If $\bar{m} < 0$, as in Figs. 2(i), (iii) and (v), there are paths which are ultimately in $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} < 0, h < 0\}$, and then the limiting behaviour as $z \downarrow$ is $\theta \uparrow \infty$, $h \uparrow 0$, hence $z_- = -\infty$.

(c) If $\bar{m} = 0$, both the preceding possibilities are open, as well as $\theta \uparrow \infty$, $h \equiv 0$, $z_- = -\infty$.

So much is fairly obvious and we omit details.[12]

[12] It was suggested in an earlier version of this paper, Foldes [1996], that a further possibilty might exist in certain cases, namely that there are paths which are *ultimately* in $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} > 0, h < 0\}$, with $h \downarrow -\infty$ and $\theta \uparrow \infty$ as $z \downarrow z_-$. This appears to be incorrect. The phase diagrams indicate that, if $\bar{m} > 0$, *every* point in $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} > 0, h < 0\}$ is the start of a backward solution passing eventually, as $z \downarrow$, into $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} < 0, h < 0\}$, then into $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} < 0, h > 0\}$ and ultimately into $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} > 0, 0 < h < \bar{f}(\theta)\}$, with limiting behaviour $h \downarrow \theta+$, $\theta \uparrow \infty$. If $\bar{m} \le 0$, *every* point in $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} > 0, h < 0\}$ is the start of a backward solution passing ultimately into $\bar{\mathcal{U}}^\triangleleft \cap \{\bar{F} < 0, h < \bar{f}(\theta) < 0\}$, with limiting behaviour $h \uparrow 0-$, $\theta \uparrow \infty$.

The following proposition sets out properties of the functions $\bar{f}$ and $\bar{g}$ which will be needed later:

Proposition 6 (Stable and unstable curves for three-parameter systems).

(i) If $\bar{\theta}_1 > 0$, the system $\bar{S} = (\bar{F}, G)$ has a saddle point at $(1, \bar{\theta}_1)$ a stable node at $(\bar{h}^-, 0)$, and an unstable node at $(\bar{h}^+, 0)$. The stable and unstable manifolds at $(1, \bar{\theta}_1)$ are represented by functions $\bar{f}$ and $\bar{g}$ defined and $\mathbf{C}^1$ for $\theta \in (0, \infty)$, (with limits $\bar{f}(0)$ and $\bar{g}(0)$ at $\theta = 0$ and $\bar{f}(\infty)$, $\bar{g}(\infty)$ at $\theta = \infty$, the limits at $\theta = 0$ being finite).

$\bar{f}$ is positive and strictly decreasing on an interval $(0, \bar{\theta}_+)$, with

$$\bar{\theta}_+ \leq \infty \quad \text{and} \quad \bar{f}(\bar{\theta}_+) = 0 \quad \text{in all cases;} \tag{3.32}$$

$$\bar{\theta}_+ = \infty \quad \text{iff} \quad \bar{m} \geq 0. \tag{3.32a}$$

Thus $\bar{f}(\infty) = +0$ if $\bar{\theta}_+ = \infty$; but $\bar{f}$ is negative on $(\bar{\theta}_+, \infty)$ if this interval is not empty, and then either $\bar{f}(\infty) = -0$ or $\bar{f}(\infty) = -\infty$ (only the former case arising if $b \leq 1$).

$\bar{g}$ is positive and strictly increasing on an interval $(\bar{\theta}_-, \infty)$, with $\bar{g}(\infty) = \infty$, and negative on $(0, \bar{\theta}_-)$ if this interval is not empty. We have

$$\bar{\theta}_- \geq 0, \quad \bar{g}(\bar{\theta}_-) \geq 0 \quad \text{and} \quad \bar{\theta}_- \cdot \bar{g}(\bar{\theta}_-) = 0 \quad \text{in all cases,} \tag{3.33}$$

$$\bar{\theta}_- = 0 \quad \text{iff} \quad \bar{m} \leq 0; \quad \bar{g}(\bar{\theta}_-) = 0 \quad \text{iff} \quad \bar{m} \geq 0. \tag{3.34}$$

It follows that, whatever the sign of $\bar{m}$, one of the functions $\bar{f}$ and $\bar{g}$ is positive on the whole of $(0, \infty)$. The following inequalities hold:

$$\bar{h}^+ = \bar{f}(0) > \bar{f}(\bar{\theta}_1) = 1, \tag{3.35}$$

$$\bar{h}^- = \bar{g}(0) < \bar{g}(\bar{\theta}_1) = 1. \tag{3.36}$$

(ii) If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, $b > 1$, then $\bar{S}$ has a saddle point at $(\bar{h}^+, 0)$ and a stable node at $(\bar{h}^-, 0)$. The stable manifold at $(\bar{h}^+ .0)$ is represented by a function $\bar{f}$ defined and $\mathbf{C}^1$ for $\theta \in [0, \infty)$. The properties of $\bar{f}$ and $\bar{\theta}_+$ are as in (i) except that (35–36) are replaced by

$$1 \geq \bar{h}^+ = \bar{f}(0) > 1/b > \bar{h}^-. \tag{3.37}$$

(iii) If $\bar{\theta}_{1/b} > 0 \geq \bar{\theta}_1$, $b < 1$, then $\bar{S}$ has a saddle point at $(\bar{h}^-, 0)$ and an unstable node at $(\bar{h}^+, 0)$. The unstable manifold is represented by a function $\bar{g}$ defined and $\mathbf{C}^1$

for $\theta \in [0, \infty)$. The properties of $\bar{g}$ and $\bar{\theta}_-$ are as in (i) except that only the case $\bar{m} < 0$, $\bar{\theta}_- = 0$ is admissible and (35–36) are replaced by

$$1 \leq \bar{h}^- = \bar{g}(0) < 1/b < \bar{h}^+ . \tag{3.38}$$[13]

(iv) *Five-parameter autonomous systems.* So far we have considered auxiliary autonomous systems $\bar{S} = (\bar{F}, G)$ with $\bar{F}$ defined (for fixed σ^2) by three parameters $(b, \bar{Q}, \bar{m})$ see (3).

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[13] Some technical remarks should be added to the preceding geometric discussion of stable/unstable manifolds of systems $\bar{S} = (\bar{F}, G)$. Given a set $\mathbf{U} \subseteq R^2$ and a stationary point $\bar{\pi}$ of $\bar{S}$, we define the stable set at $\bar{\pi}$ wrt $\mathbf{U}$ as

$$\bar{\mathcal{M}}^{\triangleright}(\bar{\pi}; \mathbf{U}) = \{\pi \in \mathbf{U} \colon \bar{\phi}^0(z; \pi) \to \bar{\pi} \text{ as } z \to \infty\},$$

where the notation entails that $\bar{\phi}^0(z; \pi)$ is defined for all $z \geq 0$. Similarly, the unstable set is

$$\bar{\mathcal{M}}^{\triangleleft}(\bar{\pi}; \mathbf{U}) = \{\pi \in \mathbf{U} \colon \bar{\phi}^0(z; \pi) \to \bar{\pi} \text{ as } z \to -\infty,$$

where $\bar{\phi}^0(z; \pi)$ is defined for $z \leq 0$.

Situations commonly considered in texts are (i) that $\mathbf{U}$ is a (sufficiently small) neighbourhood of $\bar{\pi}$ in $\Re^2$ or (ii) that $\mathbf{U} = \Re^2$. In these cases the stable/unstable sets are *called* respectively local and global stable/unstable manifolds.

These definitions are not quite what we require since for the economic interpretation of the model we are interested only in domains contained within $\{\theta \geq 0\}$. Let $\mathbf{U} = \{\theta \geq 0\}$. As Figures 2(i–v) show, there are stationary points of $\bar{S}$ situated on the boundary $\{\theta = 0\}$ of this set. Neighbourhoods of these points should properly be defined as 'neighbourhoods w.r.t. $\{\theta \geq 0\}$' or 'half-neighbourhoods', while their stable/unstable sets w.r.t. $\{\theta \geq 0\}$ are properly 'manifolds with boundary'. This said, we usually omit the qualifiers and adapt standard results.

Now restrict attention to the saddle point of $\bar{S}$, denoted $\bar{\pi}^* = (\bar{h}^*, \bar{\theta}^*)$. If $\bar{S}$ is Type 1, as in Figs.2(i–ii), the saddle point and its global stable/unstable manifold are interior to $\{\theta \geq 0\}$, so that $\mathbf{U} = \{\theta \geq 0\}$ may be replaced by $\{\theta > 0\}$ and no essential modifications of standard results are required. In this case, there are stable and unstable local manifolds, say $\bar{\mathcal{M}}^{\triangleright}_{\mathrm{loc}}$ and $\bar{\mathcal{M}}^{\triangleleft}_{\mathrm{loc}}$, which are one-dimensional differential sub-manifolds of $\mathbf{U}$. A suitable choice of co-ordinates allows $\bar{\mathcal{M}}^{\triangleright}_{\mathrm{loc}}$ to be represented in the (h, θ)-plane as the graph of a $\mathbf{C}^1$ function $h = \bar{f}(\theta)$ defined on a θ-neighbourhood of $\bar{\theta}^*$, with $\bar{f}'(\bar{\theta}^*) < 0$, hence also $\bar{f}' < 0$ for θ near $\bar{\theta}^*$. The global stable manifold $\bar{\mathcal{M}}^{\triangleright}$ is obtained as $\bigcup_{z \leq 0} \bar{\phi}^0_z(\bar{\mathcal{M}}^{\triangleright}_{\mathrm{loc}})$, i.e. as the set of points of $\mathbf{U}$ reached by 'running backwards' the solutions through points of $\bar{\mathcal{M}}^{\triangleright}_{\mathrm{loc}}$. Phase analysis shows that the representation $h = \bar{f}(\theta)$ extends to the whole interval $0 < \theta < \infty$, with $\bar{f}$ of class $\mathbf{C}^1$, and $\bar{f}' < 0$ on the interval $0 < \theta < \bar{\theta}_+$ where $\bar{f} > 0$. It follows that $\bar{\mathcal{M}}^{\triangleright}$, restricted to $\{h > 0, \theta > 0\}$, is an (embedded) $\mathbf{C}^1$ sub-manifold. Note that, for Type 1, $\bar{f}$ is properly defined only for $0 < \theta < \infty$, although sometimes we write $\bar{f}(0)$ and $\bar{f}(\infty)$ for the appropriate one-sided limits.

Similarly, $\bar{\mathcal{M}}^{\triangleleft}_{\mathrm{loc}}$ may be represented as the graph of a $\mathbf{C}^1$ function $h = \bar{g}(\theta)$ defined for θ near $\bar{\theta}^*$ with $\bar{g}'(\bar{\theta}^*) > 0$. The global unstable manifold $\bar{\mathcal{M}}^{\triangleleft}$ is obtained as $\bigcup_{z \geq 0} \bar{\phi}^0_z(\bar{\mathcal{M}}^{\triangleleft}_{\mathrm{loc}})$. Phase analysis shows that the representation $h = \bar{g}(\theta)$ extends to the interval $0 < \theta < \infty$ with $\bar{g}$ of class $\mathbf{C}^1$, and $\bar{g}' > 0$ on the interval $\bar{\theta}_- < \theta < \infty$ where $\bar{g} > 0$. So $\bar{\mathcal{M}}^{\triangleleft}$, restricted to $\{h > 0, \theta > 0\}$, is a $\mathbf{C}^1$ sub-manifold. Here again, $\bar{g}$ is properly defined only for $0 < \theta < \infty$, but we sometimes write $\bar{g}(0)$ and $\bar{g}(\infty)$ for the limits.

Turning to Type 0 cases, we leave aside the case $\bar{\pi}^* = (1, 0)$ illustrated in Fig. 5. If $\bar{S}$ is Type 0 with $b > 1$ as in Figs. 2(iii–iv), the preceding discussion of the stable manifold is essentially unaltered if the remarks about the boundary $\{\theta = 0\}$ are taken into account. Now $\bar{\mathcal{M}}^{\triangleright}$ is a manifold with boundary, having a single boundary point $\bar{\pi}^* = (\bar{h}^+, 0)$, and the representing function $\bar{f}$ is properly defined at $\theta = 0$. The unstable manifold is an interval of $\{\theta = 0\}$, so that $\bar{g}$ is undefined. Corresponding remarks apply for $\bar{\mathcal{M}}^{\triangleleft}$ if $\bar{S}$ is Type 0 with $b < 1$ and $\bar{\pi}^* = (\bar{h}^-, 0)$, with $\bar{g}(0)$ properly defined and $\bar{f}$ undefined.

In order to set bounds for the motion of S, we now consider autonomous systems $\bar{S}$ with $\bar{F}$ defined by two formulas, one of which applies for (h,θ) above and on a certain 'dividing' line, which is either $\{h=0\}$ or $\{h=1/b\}$, while the other applies for (h,θ) below and on the same line. The values of the two formulas for $\bar{F}(h,\theta)$ along the dividing line will always be equal, so that a continuous $\bar{F}$ is defined overall (but with a derivative discontinuity along the line). If the dividing line is $\{h=0\}$, each of the formulas for $\bar{F}$ will define on its sub-domain a *lower* bound for $F(h,\theta,z)$, while if the dividing line is $\{h=1/b\}$ each formula will define an *upper* bound for $F(h,\theta,z)$. Sometimes the inequalities defining bounds for F will be global (applying for all z), sometimes they will apply only for far right or only for far left values of z.

In each case considered, both formulas for $\bar{F}$ can be expressed in the form (3a) with the same b and σ^2, but possibly different $\bar{Q}$ and $\bar{m}$, and we shall tabulate appropriate values of the two latter parameters; the value of σ^2 being fixed throughout, we usually refer to systems $\bar{S}$ with $\bar{F}$ defined by two formulas as 'five-parameter' systems. Their phase analysis is similar to that of the three-parameter systems considered in 3(ii–iii) if the restrictions on the sub-domain are observed, and various details will be omitted.

The parameters $\bar{\theta}_1$, $\bar{\theta}_{1/b}$ and $\bar{R}=2\bar{m}/b\sigma^2$ may be defined for a five-parameter system as in (4–6) since they depend only on the values of $\bar{F}$ along the lines $\{h=1\}$, $\{h=1/b\}$ and $\{h=0\}$. The classification of systems as Type 1 ($\bar{\theta}_1>0$) or Type 0 ($\bar{\theta}_{1/b}>0>\bar{\theta}_1$) therefore still makes sense, and all systems $\bar{S}$ considered will be of one of these Types. When necessary to avoid ambiguity, we distinguish among parameters for different systems by means of superscripts (omitting the 'over-bar' when the system in question is sufficiently identified).

Upper and lower bounds for $F(h,\theta,z)$ are obtained from the properties of $A(z)$ and $M(z)$ — see (0.2–0.4). For brevity we write $F=F(h,\theta,z)$, $F^\wedge=F^\wedge(h,\theta)$, $S^\wedge=(F^\wedge,G)$, $F^\vee=F^\vee(h,\theta)$, $S^\vee=(F^\vee,G)$ etc. Starting with *inequalities and bounds which are valid for all* $z\in\Re$ *and for* $h\in\Re$ *and* $\theta\geq 0$, we have

$$(3.39a)\qquad F>F^\wedge\doteq\left\{\begin{array}{lll} F_{-\infty}-2\psi_0'/b\sigma^2 & \text{if } h\geq 0 & [\bar{Q}=Q+\psi_0',\quad \bar{m}=m] \\ F_\infty & \text{if } h\leq 0 & [\bar{Q}=Q,\qquad\quad \bar{m}=m] \end{array}\right\}$$

$$(3.39b)\qquad \begin{aligned} &\theta_1^\wedge=N+\psi_0'/b=n+\psi_0',\quad \theta_{1/b}^\wedge=q+\psi_0',\\ &R^\wedge\doteq -F^\wedge(0,\theta)=2m^\wedge/b\sigma^2=R_\infty,\quad m^\wedge=m.\end{aligned}$$

The expressions following the left curly brackets in (39a) define $F^\wedge$ on the domains $\{h \geq 0\}$ and $\{h \leq 0\}$, while the quantities in square brackets are the values of $\bar{Q}$ and $\bar{m}$ to be chosen in order to represent $F^\wedge$ as a function $\bar{F}$ of the type defined in (3a) on the appropriate domain. Using (1–2), it can be checked that $F^\wedge$ is continuous on $\{h = 0\}$. The values of $\theta_1^\wedge$, $\theta_{1/b}^\wedge$, $R^\wedge$ and $m^\wedge$ in (39b) are calculated as in (4–6) from the values of $F^\wedge$ along the lines $\{h = 1\}$, $\{h = 1/b\}$ and $\{h = 0\}$. We also write $h^{\wedge+}$ and $h^{\wedge-}$ for the solutions of $F^\wedge(h, 0) = 0$, calculated as in (10) using the appropriate values of $\bar{Q}$ and $\bar{m}$ from (39a). Analogous notation applies to other systems defined below. *The number $\theta_1^\wedge$ plays a special part in what follows and is denoted by ν.*

In the same way, and omitting detailed explanations, we have

(3.40a) $$F < F^\vee \doteq \left\{ \begin{array}{lll} F_\infty & \text{if } h \geq 1/b & [\bar{Q} = Q, \qquad \bar{m} = m] \\ F_{-\infty} & \text{if } h \leq 1/b & [\bar{Q} = Q + \psi_0', \quad \bar{m} = m - \psi_0'] \end{array} \right\}$$

(3.40b) $$\begin{gathered} \theta_1^\vee = n \text{ if } b > 1; \quad \theta_1^\vee = N \text{ if } b \leq 1; \quad \theta_{1/b}^\vee = q; \\ R^\vee \doteq -F^\vee(0, \theta) = 2m^\vee / b\sigma^2 = R_{-\infty}, \quad m^\vee = m - \psi_0'. \end{gathered}$$

According to our Standing Assumptions, we have $N > 0$ with $n \vee q > 0$ if $b \geq 1$ and $n > 0$ with $N \vee q > 0$ if $b \leq 1$. It follows that $\theta_1^\wedge = \nu > 0$ *so that $S^\wedge = (F^\wedge, G)$ is always a Type 1 system.* On the other hand, $S^\vee = (F^\vee, G)$ *is of the same Type (1 or 0) as S_∞*, with saddle at $(1, n)$ or at $(h_\infty^+, 0)$ *if $b \geq 1$, and of the same Type as $S_{-\infty}$*, with saddle at $(1, N)$ or at $(h_{-\infty}^-, 0)$ *if $b \leq 1$.*

The various possible combinations of $S^\wedge$ and $S^\vee$, classified according to the Type of $S^\vee$, the sign of $b - 1$ and the signs of m and $m - \psi_0'$ are illustrated (apart from certain borderline cases) in Figs. 3–4. Fig. 3 has Type 1 cases, Fig. 4 has Type 0. In each Figure, the first three diagrams relate to $b > 1$, the last three to $b < 1$, taking in turn the cases $m < 0$, $0 < m < \psi_0'$ and $m > \psi_0'$ (except that there is no Fig. 4(vi)). Cases with $b = 1$ may be assimilated to $b < 1$, Type 1. Cases with $m = 0$ or $m = \psi_0'$ will be dealt with as we go along. All diagrams are drawn with $q > 0$, even where $q \leq 0$ is consistent with our text.[14] These reservations apart, the diagrams exhaust the possibilities. Some remarks

[14] The main effect on Figs. 3–4 of setting $q > 0$ is as follows. Since $\bar{\theta}_{1/b} = q$ for each of S_∞, $S_{-\infty}$ and $S^\vee$, it follows from Prop. 4 that, for each of these systems, $\bar{\theta}_{1/b} > 0$ implies $\bar{h}^- < 1/b$ if $b \geq 1$, $\bar{h}^+ > 1/b$ if $b \leq 1$. It then follows from the definition (3.40) of $F^\vee$ that $h^{\vee-} = h_{-\infty}^- < 1/b$ if $b \geq 1$, $h^{\vee+} = h_\infty^+ > 1/b$ if $b \leq 1$. In several cases, the condition $q > 0$ follows from the Standing Assumptions and so does not impose additional restrictions on the diagrams. Thus, in Figs. 4, either S_∞ or $S_{-\infty}$ is of Type 0, so $q > 0$ is assumed. In Figs. 3, both systems are of Type 1, i.e. $n > 0$ and $N > 0$. If $b = 1$,

about the phase pictures follow.

The phase picture of $S^\wedge$ is similar to that of Type 1 systems $\bar{S}$ discussed above, making allowance for the break at $\{h = 0\}$. There is a saddle at $(1, \nu)$, an unstable node at a point $(h^{\wedge +}, 0)$, with $h^{\wedge +} > (h^+_\infty \vee h^+_{-\infty})$, and a stable node at a point $(h^{\wedge -}, 0)$ with $h^{\wedge -} \leq (h^-_\infty \wedge h^-_{-\infty})$. At the saddle there is a stable manifold $\mathcal{M}^{\wedge \triangleright}$ represented by a continuous function $h = f^\wedge(\theta)$ defined for $\theta \in (0, \infty)$, $\mathbf{C}^1$ except perhaps at a point with $f^\wedge(\theta) = 0$, satisfying $1 = f^\wedge(\nu)$, decreasing as long as $f^\wedge(\theta) > 0$ and with a left limit $h^{\wedge +} = f^\wedge(0)$. If $m \geq 0$, then $f^\wedge > 0$ on the whole axis $\theta > 0$, but if $m < 0$ then $f^\wedge = 0$ at some finite $\theta^\wedge_+$ and thereafter remains negative with $f^\wedge(\theta) \uparrow 0$ as $\theta \uparrow \infty$ as described earlier. Again, there is an unstable manifold $\mathcal{M}^{\wedge \triangleleft}$ represented by a continuous function $h = g^\wedge(\theta)$ defined for $\theta \in (0, \infty)$, $\mathbf{C}^1$ except perhaps at any point with $g^\wedge(\theta) = 0$, satisfying $1 = g^\wedge(\nu)$, increasing when $g^\wedge(\theta) > 0$ with $g^\wedge(\infty) = \infty$ and with a left limit $g^\wedge(0) = h^{\wedge -}$. If $m > 0$, then $h^{\wedge -} = h^-_\infty < 0$ and $g^\wedge$ is negative on some initial interval $(0, \theta^\wedge_-)$ and positive on $(\theta^\wedge_-, \infty)$; but if $m \leq 0$ then $h^{\wedge -} \geq 0$ and $g^\wedge$ is positive for all $\theta > 0$. *Thus in each case either $f^\wedge$ or $g^\wedge$ is positive for all $\theta > 0$*, (but both are positive only if $m = 0$). Note also the bounds which $S^\wedge$ defines for the motion S (and for every $\bar{S}$). Since $F > F^\wedge$, *the motion is always upward for $z \uparrow$ if $F^\wedge \geq 0$, in particular $F(h, \theta) > 0$ for all $\theta > 0$ when $h \geq h^{\wedge +}$*. Further, *the stable curve $\mathcal{M}^{\wedge \triangleright} = \{h = f^\wedge(\theta)\}$ forms a barrier to downward motion for S, i.e. it can be crossed only from below as $z \uparrow$ or from above as $z \downarrow$*, (where 'below' and 'above' refer to the half-spaces defined by the curve). In the same way, *the unstable curve $\mathcal{M}^{\wedge \triangleleft} = \{h = g^\wedge(\theta)\}$ can be crossed only from below as $z \uparrow$, from above as $z \downarrow$*.

Consider now $S^\vee = (F^\vee, G)$, starting with $b > 1$ and S_∞ of Type 1, so that $n > 0$ and $N = n + \psi_0'(b - 1)/b > 0$, see Figs. 3(i,ii,iii). The phase picture above the line $\{h = 1/b\}$ is obviously the same as for S_∞; thus there is a saddle at $(1, n)$, with stable and unstable curves $f^\vee$ and $g^\vee$ which coincide with f_∞ and g_∞ as long as they lie above the line $\{h = 1/b\}$. The unstable node $(h^{\vee +}, 0)$ satisfies $h^{\vee +} = h^+_\infty$, but for the stable node $(h^{\vee -}, 0)$ we have $h^{\vee -} = h^-_\infty$ only in case $h^-_\infty \geq 1/b$ (i.e. in case $q \leq 0$), otherwise $h^{\vee -} = h^-_{-\infty} < 1/b$ (see fn. 14).

The curve $f^\vee$ is defined and continuous for $\theta > 0$, and $\mathbf{C}^1$ except where $f^\vee(\theta) = 1/b$; it satisfies $f^\vee(n) = 1$ and (as a limit) $f^\vee(0) = h^{\vee +}$, and it is decreasing as long as it is

then $q = n = N > 0$. If $b > 1$, choose the parameters for S_∞ given by (3.7), and conclude from (3.6a) that $q \leq 0$ is inadmissible if $m \geq -\frac{1}{2}b\sigma^2$, as in Figs. 3(ii)–(iii). If $b < 1$, choose the parameters for $S_{-\infty}$ given by (3.8), and conclude from (3.6a) that $q \leq 0$ is inadmissible if $m \leq \psi_0' - \frac{1}{2}b\sigma^2$. In other cases it can apparently happen that $q \leq 0$ and either $h^{\vee -} = h^-_\infty \geq 1/b$ if $b > 1$, or $h^{\vee +} = h^+_{-\infty} \leq 1/b$ if $b < 1$.

positive. The behaviour of $f^\vee$ below the line $\{h = 1/b\}$ is determined by $F_{-\infty}$; thus if $m-\psi_0' \geq 0$, then $f^\vee > 0$ on the whole axis, but if $m-\psi_0' < 0$ the curve becomes negative at some $\theta_+^\vee$, etc. Again, $g^\vee$ is defined and continuous for $\theta > 0$, and $\mathbf{C}^1$ except where $g^\vee(\theta) = 1/b$; it satisfies $g^\vee(n) = 1$ and (as a limit) $g^\vee(0) = h^{\vee -}$, and it is increasing as long as it is positive. If $m - \psi_0' > 0$, it is found that $h^{\vee -} < 0$ so that $g^\vee(\theta) < 0$ on some interval $(0, \theta_-^\vee)$ and positive thereafter; but if $m - \psi_0' \leq 0$, then $g^\vee > 0$ for all $\theta > 0$. *Thus in each case either $f^\vee$ or $g^\vee$ is positive for all $\theta \in (0,\infty)$*, (but both are positive only if $m-\psi_0' = 0$). Since $F < F^\vee$, *the motion S is always downward for $z \uparrow$ if $F^\vee \leq 0$. The stable curve $\mathcal{M}^{\vee\triangleright} = \{h = f^\vee(\theta)\}$ can be crossed only from above as $z \uparrow$, and the unstable curve $\mathcal{M}^{\vee\triangleleft} = \{h = g^\vee(\theta)\}$ can be crossed only from below as $z \downarrow$.*

Taking the phase pictures for $S^\wedge$ and $S^\vee$ together (Type 1, $b > 1$), it is seen that $f^\wedge(\theta) > f^\vee(\theta)$ for all $\theta \in (0,\infty)$. (This follows, for example, because the saddle point of $S^\vee$ is in the lower half-space defined by $f^\wedge$, and $F^\wedge < F^\vee$ implies that $f^\vee$ cannot cross $f^\wedge$ from below). Consequently $\theta_+^\vee \leq \theta_+^\wedge$, with a strict inequality if either of these numbers is finite, i.e. if $m < \psi_0'$. The curves $f^\vee$ and $f^\wedge$ form the lower and upper boundaries of a 'tube'

$$\mathcal{C}^\triangleright = \{(h,\theta) : f^\vee(\theta) < h < f^\wedge(\theta), \theta > 0\}, \tag{3.41}$$

an open plane set from which paths of S can exit as $z \uparrow$ but not enter. Similarly, $g^\wedge(\theta) < g^\vee(\theta)$ for $\theta \in (0,\infty)$, so that $\theta_-^\vee \leq \theta_-^\wedge$, with a strict inequality if either of these numbers is positive, i.e. if $m > 0$. The curves $g^\wedge$ and $g^\vee$ define a 'tube'

$$\mathcal{C}^\triangleleft = \{(h,\theta) : g^\wedge(\theta) < h < g^\vee(\theta), \theta > 0\}, \tag{3.42}$$

an open plane set from which paths of S can exit as $z \downarrow$ but not enter. Moreover, any path of S which reaches the relative boundary of $\mathcal{C}^\triangleright$ or $\mathcal{C}^\triangleleft$ crosses immediately.

Systems with $b > 1$ and S_∞ of Type 0 are simpler. Now $N > 0 > n$ and $q > 0$, see Figs. 4(i,ii,iii). The phase picture for $S^\vee$ above $\{h = 1/b\}$ is the same as for S_∞, with a saddle at $(h^{\vee +}, 0)$ and $h^{\vee +} = h_\infty^+ \in (1/b, 1)$. The (one-sided) stable curve $f^\vee$ starts at $f^\vee(0) = h^{\vee +}$ and is decreasing, coinciding with f_∞ as long as the latter lies above $\{h = 1/b\}$, and its behaviour thereafter is the same as with Type 1. The functions $f^\vee$ and $f^\wedge$ define a tube $\mathcal{C}^\triangleright$ with the properties mentioned for Type 1. However, *the unstable curve now lies on the vertical axis and so fails to define a useful bound for the motion $S^\vee$*; this is the main reason why we shall introduce alternative bounds below. Since $q > 0$, the unstable node for $S^\vee$ lies below $\{h = 1/b\}$, and so $h^{\vee -} = h_{-\infty}^-$ (see fn. 14).

The discussion of cases with $b \leq 1$ is largely symmetrical with that for $b > 1$. No more need be said about $S^\wedge$. If $N > 0$, $S^\vee$ is of Type 1. This time the saddle is at $(1, N)$, the stable node is at $(h^{\vee -}, 0)$ and both points satisfy $h \leq 1/b$ (even $h < 1/b$ if $b < 1$), so the interesting part of the phase map is the same as for $S_{-\infty}$ and we have $h^{\vee -} = h^-_{-\infty} \leq 1/b$. If $q > 0$, the unstable node $(h^{\vee +}, 0)$ satisfies $h^{\vee +} = h^+_\infty > 1/b$ (see fn. 14), otherwise $h^{\vee +} = h^+_{-\infty} \leq 1/b$. The remarks about the curves $f^\vee$ and $g^\vee$, including the definitions of 'tubes', continue to apply with routine changes. In particular, $f^\vee$ and $g^\vee$ now denote curves which are respectively stable and unstable (for the forward motion) at the saddle point of $S_{-\infty}$, and they coincide with $f_{-\infty}$ and $g_{-\infty}$ as long as they remain below the line $\{h = 1/b\}$.

If $b < 1$ and $q > 0 > N$, $S^\vee$ is of Type 0; see Figs 4(iv,v). The saddle is at $(h^{\vee -}, 0)$ with $h^{\vee -} = h^-_{-\infty} \in (1, 1/b)$, and so is again in the region where the phase map is the same as for $S_{-\infty}$. Since $q > 0$ in this case, the unstable node is defined by $h^{\vee +} = h^+_\infty > 1/b$. The unstable curve $g^\vee$ starts at $g^\vee(0) = h^{\vee -}$ and is increasing, coinciding with $g_{-\infty}$ while the latter lies below $\{h = 1/b\}$ and becoming unbounded thereafter. Also $R^\vee = R_{-\infty}$, which for $b < 1$ and $S_{-\infty}$ of Type 0 must be negative, so that only cases with $m - \psi_0' < 0$ are admissible here. We have $g^\wedge < g^\vee$ and these curves form a tube $\mathcal{C}^\triangleleft$ as before. *This time it is the stable curve which lies on the vertical axis.*

Sharper bounds for F can be defined at far left or far right values of z. Given any $\delta > 0$, $\delta < \psi_0'$, we can choose z^δ so far left that

(3.43) $$0 < \psi_0' - A(z) < \psi_0' - M(z) < \delta < \psi_0' \quad \text{for } z \in (-\infty, z^\delta],$$

and then, for these values of z and for $h \in \Re$ and $\theta \geq 0$, we have

(3.44a)
$$F > F^{\wedge\delta} \doteq \begin{cases} F_{-\infty} - 2\delta/b\sigma^2 & \text{if } h \geq 0 \quad [\bar{Q} = Q + \psi_0', \qquad \bar{m} = m - \psi_0' + \delta] \\ F_{-\infty} - 2\delta(1 - bh)/b\sigma^2 & \text{if } h \leq 0 \quad [\bar{Q} = Q + \psi_0' - \delta, \quad \bar{m} = m - \psi_0' + \delta] \end{cases}$$

(3.44b) $$\theta_1^{\wedge\delta} = N + \delta/b, \quad \theta_{1/b}^{\wedge\delta} = q + \delta,$$
$$R^{\wedge\delta} \doteq -F^{\wedge\delta}(0, \theta) = 2m^{\wedge\delta}/b\sigma^2 = R_{-\infty} + 2\delta/b\sigma^2, \quad m^{\wedge\delta} = m - \psi_0' + \delta.$$

Also,

(3.45a)

$$F < F^{\vee\delta} \doteq \left\{ \begin{array}{lll} F_{-\infty} + 2\delta(bh-1)/b\sigma^2 & \text{if } h \geq 1/b & [\bar{Q} = Q + \psi_0' - \delta, \quad \bar{m} = m - \psi_0' + \delta] \\ F_{-\infty} & \text{if } h \leq 1/b & [\bar{Q} = Q + \psi_0', \qquad \bar{m} = m - \psi_0'] \end{array} \right\}$$

(3.45b)
$$\theta_1^{\vee\delta} = N - \delta(b-1)/b \text{ if } b > 1; \quad \theta_1^{\vee\delta} = N \text{ if } b \leq 1; \quad \theta_{1/b}^{\vee\delta} = q;$$
$$R^{\vee\delta} \doteq -F^{\vee\delta}(0,\theta) = 2m^{\vee\delta}/b\sigma^2 = R_{-\infty}, \quad m^{\vee\delta} = m - \psi_0'.$$

Unless otherwise stated, the resulting systems $S^{\wedge\delta} = (F^{\wedge\delta}, G)$ *and* $S^{\vee\delta} = (F^{\vee\delta}, G)$ *will be considered only for cases with* $b > 1$, *hence* $N > 0$, and then for small $\delta > 0$ the phase pictures look roughly like that for $S_{-\infty}$, with saddle points slightly to the right and left of the saddle of $S_{-\infty}$ at $(1, N)$ and unstable curves $g^{\wedge\delta}$ and $g^{\vee\delta}$ slightly below and above $g_{-\infty}$; see Figs. 3(i,ii,iii). More precisely, comparison among the systems $S^{\wedge}$, $S^{\wedge\delta}$, $S_{-\infty}$, $S^{\vee\delta}$, $S^{\vee}$ shows that

(3.46a)
$$F^{\wedge} < F^{\wedge\delta} < F_{-\infty} \leq F^{\vee\delta} \leq F^{\vee}$$

and that the corresponding values of $\bar{\theta}_1$ decrease along the sequence — explicitly,

(3.46b)
$$\theta_1^{\wedge} > \theta_1^{\wedge\delta} > \theta_1(-\infty) > \theta_1^{\vee\delta} > \theta_1^{\vee}, \quad \text{i.e.}$$
$$\nu = N + \psi_0'/b > N + \delta/b > N > N - \delta(b-1)/b > N - \psi_0'(b-1)/b = n,$$

see (39), (44), (8), (45), (40). These values of θ, except perhaps $\theta_1^{\vee}$, are positive for small δ, and if positive they define the positions of corresponding saddle points along $\{h = 1\}$. In particular, $S^{\wedge\delta}$, $S_{-\infty}$ *and* $S^{\vee\delta}$ *will be of Type 1 in all cases with* $b > 1$ *and small* $\delta > 0$. Next, $h^-_{-\infty} < 1 < h^+_{-\infty}$ because $N > 0$ (Prop. 4), and in case $q > 0$ we also have $h^-_{-\infty} < 1/b < h^+_{-\infty}$; the corresponding inequalities also hold with $h^{\pm}_{-\infty}$ replaced by $h^{\wedge\delta\pm}$ or by $h^{\vee\delta\pm}$. Corresponding to $g_{-\infty}(0) = h^-_{-\infty}$ we have $g^{\wedge\delta}(0) = h^{\wedge\delta-}$ and $g^{\vee\delta}(0) = h^{\vee\delta-}$, with $g^{\wedge\delta}(0) < g_{-\infty}(0) \leq g^{\vee\delta}(0)$; also $g^{\wedge}(0) < g^{\wedge\delta}(0)$. Taking into account (46a–b), it follows easily that $g^{\wedge} < g^{\wedge\delta} < g_{-\infty} \leq g^{\vee\delta}$ for $\theta \in [0, \infty)$. By Prop. 6(i) with $\bar{S} = S_{-\infty}$, $g_{-\infty}$ is positive on some interval $(\theta_-(\infty), \infty)$, where $\theta_- = \theta_-(-\infty)$. Further, if $m - \psi_0' < 0$, then $R_{-\infty} < 0$ and $g_{-\infty}$ is positive and increasing on $(0, \infty)$; and so for small δ both $R^{\wedge\delta}$ and $R^{\vee\delta}$ are negative and both $g^{\wedge\delta}$ and $g^{\vee\delta}$ are positive and increasing on $(0, \infty)$. On the other hand, if $m - \psi_0' \geq 0$, then $R_{-\infty} \geq 0$ and $g_{-\infty}$ is positive only on an interval (θ_-, ∞) and is increasing there; and so for small δ both $R^{\wedge\delta}$

and $R^{\vee\delta}$ are ≥ 0, while $g^{\wedge\delta}$ and $g^{\vee\delta}$ are positive only on intervals $(\theta_-^{\wedge\delta}, \infty)$ and $(\theta_-^{\vee\delta}, \infty)$ respectively and are increasing on these intervals. (The weak inequalities allow for the case $m = \psi_0'$). Clearly $\theta_-^{\vee\delta} \leq \theta_- \leq \theta_-^{\wedge\delta}$, and $\theta_-^{\vee\delta} < \theta_-^{\wedge\delta}$ if one of these numbers is positive, i.e. if $m - \psi_0' \geq 0$. In any case, $g_-^{\wedge\delta}(\infty) = g_-^{\vee\delta}(\infty) = \infty$. (These results also hold for $b \leq 1$ in case $N > 0$ so that $S_{-\infty}$ is Type 1, except that the saddle point of $S^{\vee\delta}$ is then at $(1, N)$.)

Referring now to the discussion of $S^\wedge$ and $S^\vee$, we recall that $f^\wedge$ and $f^\vee$ are both positive on the whole axis $(0, \infty)$ in case $m - \psi_0' \geq 0$, whereas for $m - \psi_0' < 0$ the functions are positive only on intervals $(0, \theta_+^\wedge)$ and $(0, \theta_+^\vee)$ with $0 < \theta_+^\vee < \theta_+^\wedge \leq \infty$. Thus, *in all cases with* $b \geq 1$, *one of the pairs of functions* $f^\wedge$, $f^\vee$ *and* $g^{\wedge\delta}$, $g^{\vee\delta}$ *is always positive for all* $\theta \in (0, \infty)$ *and small* δ. Henceforth it is assumed without special mention that (43) and other properties requiring a small δ are satisfied.

Since $f^\wedge > f^\vee$ and $g^\wedge < g^{\wedge\delta} < g^{\vee\delta}$, it follows from

$$\theta_1^\wedge = \nu \quad \text{and} \quad f^\wedge(\nu) = g^\wedge(\nu) = 1 \tag{3.46c}$$

that

$$f^\vee(\nu) < f^\wedge(\nu) < g^{\wedge\delta}(\nu) < g^{\vee\delta}(\nu). \tag{3.47}$$

We also know that, if $b > 1$ and $n > 0$, so that $S^\vee$ and $S^{\vee\delta}$ are both of Type 1, then $f^\vee(0) = h^{\vee+}$ and $g^{\vee\delta}(0) = h^{\vee\delta-}$ lie on opposite sides of $\{h = 1\}$, hence

$$f^\wedge(0) > f^\vee(0) > 1 > g^{\vee\delta}(0) > g^{\wedge\delta}(0) \quad \text{if } n > 0. \tag{3.48}$$

On the other hand, if $q > 0 \geq n$, so that $S^\vee$ is of Type 0, then $\theta_{1/b}^\vee = \theta_{1/b}^{\vee\delta} = q$ implies $f^\vee(0) = h^{\vee+} > 1/b$ and $g^{\vee\delta}(0) = h^{\vee\delta-} < 1/b$ (Prop. 4), hence

$$f^\wedge(0) > f^\vee(0) > 1/b > g^{\vee\delta}(0) > g^{\wedge\delta}(0) \quad \text{if } q > 0 \geq n. \tag{3.49}$$

Another point to note is that, since $F^{\wedge\delta} < F < F^{\vee\delta}$ for $z \leq z^\delta$, the motion of S as $z \downarrow$ is always downward when $F^{\wedge\delta} \geq 0$ and upward when $F^{\vee\delta} \leq 0$. *Further, for* $z \leq z^\delta$ *the curve* $g^{\wedge\delta}$ *can be crossed by a path of* S *only from above as* $z \downarrow$ *and* $g^{\vee\delta}$ *can be crossed only from below. Thus the curves define a 'tube'*

$$\mathcal{C}^{\lhd\delta} = \{(h, \theta) \colon g^{\wedge\delta}(\theta) < h < g^{\vee\delta}(\theta), \quad \theta > 0\}, \tag{3.50}$$

(*or simply* $\mathcal{C}^\delta$) *from which paths of* S *can exit as* $z\downarrow$, $z \le z^\delta$, *but not enter.* These statements are illustrated in Figs. 3–4(i)–(iii).

Now consider far right values of z. Given $\rho > 0$, $\rho < \psi_0'$, one can choose z^ρ so that

$$0 < M(z) < A(z) < \rho < \psi_0' \quad \text{for } z \in [z^\rho, \infty), \tag{3.51}$$

and then, for these values of z and for $h \in \Re$ and $\theta \ge 0$, we have

$$F > F^{\wedge\rho} \doteq \left\{ \begin{array}{lll} F_\infty - 2\rho h/\sigma^2 & \text{if } h \ge 0 & [\bar{Q} = Q + \rho, \quad \bar{m} = m] \\ F_\infty & \text{if } h \le 0 & [\bar{Q} = Q, \qquad \bar{m} = m] \end{array} \right\} \tag{3.52a}$$

$$\begin{aligned} &\theta_1^{\wedge\rho} = n + \rho, \quad \theta_{1/b}^{\wedge\rho} = q + \rho(b-1)/b, \\ &R^{\wedge\rho} \doteq -F^{\wedge\rho}(0,\theta) = 2m^{\wedge\rho}/b\sigma^2 = R_\infty, \quad m^{\wedge\rho} = m. \end{aligned} \tag{3.52b}$$

Also,

$$F < F^{\vee\rho} \doteq \left\{ \begin{array}{lll} F_\infty & \text{if } h \ge 1/b & [\bar{Q} = Q, \qquad \bar{m} = m] \\ F_\infty + 2\rho(1-bh)/b\sigma^2 & \text{if } h \le 1/b & [\bar{Q} = Q + \rho, \quad \bar{m} = m - \rho] \end{array} \right\} \tag{3.53a}$$

$$\begin{aligned} &\theta_1^{\vee\rho} = n \text{ if } b \ge 1; \quad \theta_1^{\vee\rho} = n + \rho(b-1)/b \text{ if } b \le 1; \quad \theta_{1/b}^{\vee\rho} = q; \\ &R^{\vee\rho} \doteq -F^{\vee\rho}(0,\theta) = 2m^{\vee\rho}/b\sigma^2 = R_\infty - 2\rho/b\sigma^2, \quad m^{\vee\rho} = m - \rho. \end{aligned} \tag{3.53b}$$

Unless otherwise stated, the *resulting systems* $S^{\wedge\rho} = (F^{\wedge\rho}, G)$ *and* $S^{\vee\rho} = (F^{\vee\rho}, G)$ *will be considered only for cases with* $b \le 1$, *hence* $n > 0$, and then for small $\rho > 0$ the phase pictures look roughly like that for S_∞, with saddle points slightly to the right and left of the saddle of S_∞ at $(1, n)$ and stable curves $f^{\wedge\rho}$ and $f^{\vee\rho}$ slightly above and below f_∞; see Figs. 3(iv,v,vi). More precisely, comparison among the systems $S^\wedge$, $S^{\wedge\rho}$, S_∞, $S^{\vee\rho}$, $S^\vee$ shows that

$$F^\wedge \le F^{\wedge\rho} \le F_\infty \le F^{\vee\rho} \tag{3.54a}$$

and that the corresponding values of $\bar\theta_1$ decrease along the sequence — explicitly,

$$\text{(3.54b)} \qquad \begin{aligned} &\theta_1^\wedge > \theta_1^{\wedge\rho} > \theta_1(\infty) > \theta_1^{\vee\rho} > \theta_1^\vee, \quad \text{i.e.} \\ &\nu = n + \psi_0' > n + \rho > n > n + \rho(b-1)/b = N - (\psi_0' - \rho)(b-1)/b > N, \end{aligned}$$

see (39), (52), (7), (53), (40). These numbers, except perhaps $\theta_1^\vee$, are positive for small $\rho > 0$, and if positive they define the positions of corresponding saddle points along $\{h = 1\}$. In particular, $S^{\wedge\rho}$, S_∞ *and* $S^{\vee\rho}$ *will be of Type 1 in all cases with* $b \le 1$ *and small* $\rho > 0$. Next, $h_\infty^- < 1 < h_\infty^+$ because $n > 0$ (Prop. 4), and in case $q > 0$ we also have $h_\infty^- < 1/b < h_\infty^+$; the corresponding inequalities also hold with $h_\infty^\pm$ replaced by $h^{\wedge\rho\pm}$ or by $h^{\vee\rho\pm}$. Corresponding to $f_\infty(0) = h_\infty^+$ we have $f^{\wedge\rho}(0) = h^{\wedge\rho+}$ and $f^{\vee\rho}(0) = h^{\vee\rho+}$, with $f^{\wedge\rho}(0) > f_\infty(0) \ge f^{\vee\rho}(0)$; also $f^\wedge(0) > f^{\wedge\rho}(0)$. Taking into account (54a–b), it follows that $f^\wedge > f^{\wedge\rho} > f_\infty \ge f^{\vee\rho}$ for $\theta \in [0,\infty)$. By Prop. 6(i) with $\bar S = S_\infty$, f_∞ is positive on some interval $(0,\theta_+)$ with $\theta_+ = \theta_+(\infty)$. Further, if $m > 0$ then $R_\infty > 0$ and f_∞ is positive and decreasing on $(0,\infty)$; and so for small ρ both $R^{\wedge\rho}$ and $R^{\vee\rho}$ are positive and both $f^{\wedge\rho}$ and $f^{\vee\rho}$ are positive and decreasing on $(0,\infty)$. On the other hand, if $m \le 0$, then $R_\infty \le 0$ and f_∞ is positive only on an interval $(0,\theta_+)$ and is decreasing there; and so for small ρ both $R^{\wedge\rho}$ and $R^{\vee\rho}$ are ≤ 0, while $f^{\wedge\rho}$ and $f^{\vee\rho}$ are positive only on intervals $(0,\theta_+^{\wedge\rho})$ and $(\theta_+^{\vee\rho},\infty)$ respectively and are decreasing on these intervals. (Here the weak inequalities allow for $m = 0$). Clearly $\theta_+^{\vee\rho} \le \theta_+ \le \theta_+^{\wedge\rho} \le \infty$, and $\theta_+^{\vee\rho} < \theta_+^{\wedge\rho}$ if one of these numbers is finite, i.e. if $m \le 0$. In any case, $f^{\wedge\rho}(\theta_+^{\wedge\rho}) = f^{\vee\rho}(\theta_+^{\vee\rho}) = 0$. (These results also hold for $b \ge 1$ and small ρ in case $n > 0$ so that S_∞ is Type 1, except that the saddle point of $S^{\vee\rho}$ is then at $(1,n)$.)

Now recall that $g^\wedge$ and $g^\vee$ are both positive on the whole axis $(0,\infty)$ in case $m \le 0$, whereas for $m > 0$ the functions are positive only on intervals $(\theta_-^\wedge,\infty)$ and $(\theta_-^\vee,\infty)$ with $0 \le \theta_-^\vee < \theta_-^\wedge < \infty$. Thus, *in all cases with* $b \le 1$, *one of the pairs of functions* $f^{\wedge\rho}$, $f^{\vee\rho}$ *and* $g^\wedge$, $g^\vee$ *is always positive for all* $\theta \in (0,\infty)$ *and small* ρ. Henceforth it is assumed without special mention that (51) and other properties requiring a small ρ are satisfied.

Clearly (46c) remains in force for $b \le 1$, and (47–9) remain valid if n, $f^\wedge$, $f^\vee$, $g^{\vee\delta}$, $g^{\wedge\delta}$ are replaced therein by N, $f^{\wedge\rho}$, $f^{\vee\rho}$, $g^\vee$, $g^\wedge$. Also, since $F^{\wedge\rho} < F < F^{\vee\rho}$ for $z \ge z^\rho$, the motion of S as $z\uparrow$ is then always upward when $F^{\wedge\rho} \ge 0$ and downward when $F^{\vee\rho} \le 0$. *Further, for* $z \ge z^\rho$ *the curve* $f^{\wedge\rho}$ *can be crossed by a path of* S *only from below as* $z\uparrow$, *and* $f^{\vee\rho}$ *only from above. Thus the curves define a 'tube'*

$$\text{(3.55)} \qquad \mathcal{C}^{\triangleright\rho} = \{(h,\theta)\colon f^{\vee\rho}(\theta) < h < f^{\wedge\rho}(\theta), \quad \theta > 0\}$$

(or simply $\mathcal{C}^\rho$) from which paths of S can exit as $z\uparrow$, $z \geq z^\rho$, but not enter. These statements are illustrated in Figs. 3(iv)–(vi) and 4(iv)-(v).

Collecting results from the preceding discussion, and referring to Figs. 3–4, we state

PROPOSITION 7 (Stable and unstable curves for five-parameter systems $\bar{S}$).

(i) Let $b > 1$, $N > 0$ and $n \vee q > 0$ and choose $\delta > 0$ as in (3.43) ff. The functions $f^\wedge$, $f^\vee$ and $g^{\vee\delta}$, $g^{\wedge\delta}$ are defined and continuous for $\theta \in (0,\infty)$ and are $\mathbf{C}^1$ apart from isolated derivative discontinuitues, (with finite limits at $\theta = 0$ which in appropriate cases are also values of the functions). They satisfy

(3.56) $$f^\wedge(\theta) > f^\vee(\theta), \quad g^{\vee\delta}(\theta) > g^{\wedge\delta}(\theta) \quad \text{for } \theta \in [0,\infty)$$

and the inequalities (3.47–49).

The functions $f^\wedge$, $f^\vee$ are positive and strictly decreasing on intervals $(0,\theta_+^\wedge)$, $(0,\theta_+^\vee)$, and negative on intervals $(\theta_+^\wedge,\infty)$, $(\theta_+^\vee,\infty)$ if these intervals are not empty. Limits at $\theta = \infty$ are as in Prop. 6(i). We have

(3.57) $$0 < \theta_+^\vee \leq \theta_+^\wedge \leq \infty \quad \text{and} \quad f^\vee(\theta_+^\vee) = f^\wedge(\theta_+^\wedge) = 0 \quad \text{in all cases;}$$

(3.57a) $$\theta_+^\vee = \infty \text{ iff } m \geq \psi_0'; \quad \theta_+^\wedge = \infty \text{ iff } m \geq 0.$$

The functions $g^{\vee\delta}$, $g^{\wedge\delta}$ are positive and strictly increasing on intervals $(\theta_-^{\vee\delta},\infty)$, $(\theta_-^{\wedge\delta},\infty)$, with $g^{\vee\delta}(\infty) = g^{\wedge\delta}(\infty) = \infty$, and negative on $(0,\theta_-^{\vee\delta})$, $(0,\theta_-^{\wedge\delta})$ if these intervals are not empty. We have

(3.58) $$0 \leq \theta_-^{\vee\delta} \leq \theta_-^{\wedge\delta} < \infty \quad \text{and} \quad \theta_-^{\vee\delta} \cdot g^{\vee\delta}(\theta_-^{\vee\delta}) = \theta_-^{\wedge\delta} \cdot g^{\wedge\delta}(\theta_-^{\wedge\delta}) = 0 \quad \text{in all cases;}$$

(3.59) $$\theta_-^{\vee\delta} = 0 \text{ iff } m \leq \psi_0'; \quad \theta_-^{\wedge\delta} = 0 \text{ iff } m < \psi_0';$$
$$g^{\vee\delta}(\theta_-^{\vee\delta}) = 0 \text{ iff } m \geq \psi_0'; \quad g^{\wedge\delta}(\theta_-^{\wedge\delta}) = 0 \text{ iff } m \geq \psi_0'.$$

(ii) Let $b \leq 1$, $n > 0$, $N \vee q > 0$ and choose $\rho > 0$ as in (3.51) ff. The assertions under (i) remain valid if

(3.60) $$N,\ n,\ f^\wedge,\ f^\vee,\ g^{\wedge\delta},\ g^{\vee\delta},\ \theta_+^\wedge,\ \theta_+^\vee,\ \theta_-^{\wedge\delta},\ \theta_-^{\vee\delta}$$

are replaced by

(3.61) $$n,\ N,\ f^{\wedge\rho},\ f^{\vee\rho},\ g^{\wedge},\ g^{\vee},\ \theta_+^{\wedge\rho},\ \theta_+^{\vee\rho},\ \theta_-^{\wedge},\ \theta_-^{\vee},$$

including replacements in (3.47–49), with the following exceptions: In place of (57a) and (59) we have

(3.62) $$\theta_+^{\vee\rho} = \infty \text{ iff } m > 0; \quad \theta_+^{\wedge\rho} = \infty \text{ iff } m \geq 0;$$

(3.63) $$\theta_-^{\vee} = 0 \text{ iff } m \leq \psi_0'; \quad \theta_-^{\wedge} = 0 \text{ iff } m \leq 0;$$
$$g^{\vee}(\theta_-^{\vee}) = 0 \text{ iff } m \geq \psi_0'; \quad g^{\wedge}(\theta_-^{\wedge}) = 0 \text{ iff } m \geq 0.$$

If $q > 0 \geq n$, then only cases with $m - \psi_0' < 0$ are admissible.

(iii) Condition (3.46c) holds in all cases.[15]

[15] Prop. 7(i) as stated involves the functions $f^{\wedge}$and $f^{\vee}$, but these can be replaced with minor changes by $f^{\wedge\delta}$ and $f^{\vee\delta}$, yielding a 'Tube' $\mathcal{C}^{\triangleright\delta}$ which is narrower than $\mathcal{C}^{\triangleright}$, hence defining more precise bounds for the function $f(\theta)$ obtained in Prop. 12(α) below. Similarly Prop. 7(ii) as stated involves $g^{\wedge}$ and $g^{\vee}$, which can be replaced with minor changes by $g^{\wedge\rho}$ and $g^{\vee\rho}$, yielding a 'Tube' $\mathcal{C}^{\triangleleft\rho}$ which is narrower than $\mathcal{C}^{\triangleleft}$, hence defining more precise bounds for the function $g(\theta)$ obtained in Prop. 13(β) below.

4 Existence Proofs

By virtue of Prop. 5, Theorem 4A can be restated as follows:

Theorem 4B (Existence of 'Star' Solutions).

In all cases consistent with the Standing Assumptions, the system $S = (F, G)$ defined by (0.1) has a 'star' solution, i.e., a solution $\phi^* = (h^*, \theta^*) = (h^*(z), \theta^*(z); z \in \Re)$ which converges as $z \to \infty$ to the saddle point π^*_∞ of S_∞ and as $z \to -\infty$ to the saddle point $\pi^*_{-\infty}$ of $S_{-\infty}$.

We recall that the saddle point of S_∞ is at $(1, n)$ if $n > 0$ (in particular, if $b \leq 1$), and at $(h^+_\infty, 0)$ if $b \geq 1$, $q > 0 \geq n$. The saddle point of $S_{-\infty}$ is at $(1, N)$ if $N > 0$ (in particular, if $b > 1$), and at $(h^-_{-\infty}, 0)$ if $b \leq 1$, $q > 0 \geq N$.

Let $\mathbf{U}$ be $\{\theta > 0\}$ or $\{\theta \geq 0\}$. We say that a solution $\phi(z) = (h(z), \theta(z))$ of S which is defined on some right unbounded interval $I^\triangleright$ and which converges to π^*_∞ as $z \to \infty$ is a *forward special solution (f.s.s.)* relative to $\mathbf{U}$; then a point $(\pi_\Diamond, z_\Diamond) = (h_\Diamond, \theta_\Diamond, z_\Diamond) \in \mathbf{U} \times I^\triangleright$ which satisfies $\phi(z_\Diamond) = \pi_\Diamond$ is a *f.s. start.* Similarly, a solution ϕ which is defined on some left unbounded interval $I^\triangleleft$ and which converges to $\pi^*_{-\infty}$ as $z \to -\infty$ is a *backward special solution (b.s.s.)* relative to $\mathbf{U}$; then a point $(\pi_\Diamond, z_\Diamond) \in \mathbf{U} \times I^\triangleleft$ which satisfies $\phi(z_\Diamond) = \pi_\Diamond$ is a *b.s. start.* Thus a solution of S defined for all $z \in \Re$ is a star solution iff it is both a f.s.s. and a b.s.s. The object of the present Section is to prove the existence (as well as the uniqueness[1]) of such a solution in each case consistent with the Standing Assumptions, as well as to establish some related results on the structure of S.

We begin with some further notation for solutions of S. Given $\mathbf{U}$, the set of all f.s. starts relative to $\mathbf{U}$ is denoted $\mathcal{M}^\triangleright(\mathbf{U} \times \Re)$ or simply $\mathcal{M}^\triangleright$, and the set of all b.s. starts is $\mathcal{M}^\triangleleft(\mathbf{U} \times \Re)$ or simply $\mathcal{M}^\triangleleft$. Thus

(4.1a)
$$\mathcal{M}^\triangleright = \{(\pi_\Diamond, z_\Diamond) = (h_\Diamond, \theta_\Diamond, z_\Diamond) \in \mathbf{U} \times \Re\colon \phi(z; \pi_\Diamond, z_\Diamond) \to \pi^*_\infty, \quad z_\Diamond \leq z \uparrow \infty\},$$
(4.1b)
$$\mathcal{M}^\triangleleft = \{(\pi_\Diamond, z_\Diamond) = (h_\Diamond, \theta_\Diamond, z_\Diamond) \in \mathbf{U} \times \Re\colon \phi(z; \pi_\Diamond, z_\Diamond) \to \pi^*_{-\infty}, \quad z_\Diamond \geq z \downarrow -\infty\}.$$

[1] The uniqueness of a 'star' solution will follow directly from the proof of Theorem 4; see in particular Remark 3 below.

The section of $\mathcal{M}^{\triangleright}$ at a fixed $z_{\Diamond}$ is written $\mathcal{M}^{\triangleright}(z_{\Diamond})$, and the section at a fixed $(\theta_{\Diamond}, z_{\Diamond})$ is $\mathcal{M}^{\triangleright}(\theta_{\Diamond}, z_{\Diamond})$; for example

$$\mathcal{M}^{\triangleright}(z_{\Diamond}) = \{\pi_{\Diamond} \in \mathbf{U}\colon \phi(z; \pi_{\Diamond}, z_{\Diamond}) \to \pi^*_{\infty}, \quad z_{\Diamond} \le z \uparrow \infty\}. \tag{4.1c}$$

Analogous notation for sections of $\mathcal{M}^{\triangleleft}$.

We also define the set $\mathcal{B}^{\triangleright}$ of starts $(\pi_{\Diamond}, z_{\Diamond}) \in \mathbf{U} \times \Re$ of forward solutions of S which converge as $z \uparrow \infty$ to the stable node $(h^-_{\infty}, 0)$ of S_{∞} and the set $\mathcal{B}^{\triangleleft}$ of starts of backward solutions of S which converge as $z \downarrow -\infty$ to the unstable node $(h^+_{-\infty}, 0)$ of $S_{-\infty}$. Further, $\mathcal{U}^{\triangleright}$ ($\mathcal{U}^{\triangleleft}$) will denote the sets of forward (backward) starts of solutions of S which become unbounded as $z \uparrow z_+$ ($z \downarrow z_-$). Notation for sections as above.

Usually we choose $\mathbf{U} = \{\theta > 0\}$ in this Section; some remarks on solutions with $\theta = 0$ will be made as we go along.

We further use the following notation for subsets of $\{\theta > 0\} = \{(h, \theta)\colon \theta > 0\}$ which extends definitions stated in Section 3 (cf. 3.24–25, 3.41–42, 3.50, 3.55):

$$\mathcal{U}^{\wedge\triangleright} = \{h > f^{\wedge}(\theta)\}, \quad \mathcal{M}^{\wedge\triangleright} = \{h = f^{\wedge}(\theta)\}; \tag{4.2a}$$

also, if $f^{\vee}(\theta)$ is defined for all $\theta > 0$ — in particular, if $b > 1$ —

$$\mathcal{B}^{\vee\triangleright} = \{h < f^{\vee}(\theta)\}, \quad \mathcal{M}^{\vee\triangleright} = \{h = f^{\vee}(\theta)\}, \quad \mathcal{C}^{\triangleright} = \{f^{\vee}(\theta) < h < f^{\wedge}(\theta)\}, \tag{4.2b}$$

cf. (3.24) and (3.41). If $f^{\wedge}$ and $f^{\vee}$ are replaced by $f^{\wedge\rho}$ and $f^{\vee\rho}$ — see(3.51) ff — we denote the corresponding sets by $\mathcal{U}^{\wedge\rho}$, $\mathcal{M}^{\wedge\rho}$, $\mathcal{B}^{\vee\rho}$, $\mathcal{M}^{\vee\rho}$, $\mathcal{C}^{\rho}$, (omitting the superscript $\triangleright$). Again, we write

$$\mathcal{U}^{\wedge\triangleleft} = \{h < g^{\wedge}(\theta)\}, \quad \mathcal{M}^{\wedge\triangleleft} = \{h = g^{\wedge}(\theta)\}; \tag{4.2c}$$

also, if $g^{\vee}$ is defined — in particular, if $b \le 1$ —

$$\mathcal{B}^{\vee\triangleleft} = \{h > g^{\vee}(\theta)\}, \quad \mathcal{M}^{\vee\triangleleft} = \{h = g^{\vee}(\theta)\}, \quad \mathcal{C}^{\triangleleft} = \{g^{\wedge}(\theta) < h < g^{\vee}(\theta)\}, \tag{4.2d}$$

cf. (3.25) and (3.42). If $g^{\wedge}$ and $g^{\vee}$ are replaced by $g^{\wedge\delta}$ and $g^{\vee\delta}$ — see(3.43) ff. — we write $\mathcal{U}^{\wedge\delta}$, $\mathcal{M}^{\wedge\delta}$, $\mathcal{B}^{\vee\delta}$, $\mathcal{M}^{\vee\delta}$, $\mathcal{C}^{\delta}$ (omitting $\triangleleft$). Conventions like those stated in fn.13 of S.3 apply.[2]

[2]Obviously the sets $\mathcal{M}^{\triangleright}$ and $\mathcal{M}^{\triangleleft}$ as defined in (4.1) are analogous to the usual 'stable' and 'unstable'

We further recall that, for a set $\mathcal{A} \subseteq \mathbf{U} = \{\theta > 0\}$, the relative closure of $\mathcal{A}$ in $\mathbf{U}$ is written $[\![\mathcal{A}]\!]$, see S.3, fn 1.

We next consider various properties of solutions of S.

PROPOSITION 8 (Bounds for Solutions of S).

Let (h, θ) be a solution of S defined and finite for $z \in [z_a, z_b]$, $-\infty \leq z_a < z_b \leq \infty$, with $\theta(z) > 0$, (values at $\pm\infty$, if relevant, being defined as limits).

(i) If $h(z_a) > 0$ and $h(z_b) > 0$, then $h(z) > 0$ on $[z_a, z_b]$.

(ii) If $\theta(z_a) < \nu$ and $\theta(z_b) < \nu$, then $\theta(z) \leq \nu$ on $[z_a, z_b]$.

PROOF (i) We first show that $h(z) \geq 0$ on $[z_a, z_b]$. If this were false, there would be $z_a < z_\alpha < z_\beta < z_b$ such that $h(z)$ passes from positive to non-positive values at z_α as $z \uparrow$ and from negative to non-negative values at z_β. This implies

$$2[M(z_\alpha) - m]/b\sigma^2 = F[0, \theta(z_\alpha), z_\alpha] \leq 0 \leq F[0, \theta(z_\beta), z_\beta] = 2[M(z_\beta) - m]/b\sigma^2$$

hence $M(z_\alpha) \leq M(z_\beta)$, contrary to the assumption that $M(z)$ is strictly decreasing. There remains the possibility that $h(z) \geq 0$ on $[z_a, z_b]$ but $h(z_0) = 0$ for some $z_0 \in (z_a, z_b)$. Then h has a minimum at z_0, hence $h'(z_0) = F[h(z_0), \theta(z_0), z_0] = 0$ and $h''(z_0) = (dF/dz)_{Z=Z_0} \geq 0$. Evaluating these conditions one gets $M'(z_0) \geq 0$, again contrary to assumption.$\|$

(ii) If the assertion were false, there would be $z_a < z_\alpha < z_\beta < z_b$ such that $\theta(z)$ crosses the line $\{\theta = \nu\}$ from left to right at $z = z_\alpha$ and from right to left at $z = z_\beta$. This implies $\theta'(z_\alpha) = h(z_\alpha) - 1 \geq 0$, $\theta'(z_\beta) = h(z_\beta) - 1 < 0$, (taking into account that $F[1, \nu, z] > F^\wedge[1, \nu] = 0$, so that the case $h(z_\beta) - 1 = 0$ is ruled out). But then $h(z_\alpha) \geq 1 = f^\wedge(\nu) = f^\wedge(\theta(z_\alpha)) = f^\wedge(\theta(z_\beta)) > h(z_\beta)$, and since no passage is possible from $\{h \geq f^\wedge(\theta)\}$ to $\{h < f^\wedge(\theta)\}$ as $z \uparrow$ we have a contradiction.$\|$

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manifolds for stationary points of differentiable dynamical systems, but for clarity we reserve the terms 'stable' and 'unstable' for autonomous systems; cf. fn.13 and Prop.7 above. Bear in mind also that π^*_∞ and $\pi^*_{-\infty}$ are saddle points of S_∞ and $S_{-\infty}$ (not of S), whereas our 'special' solutions are solutions of S, so that stable/unstable manifold theorems as usually stated do not apply to $\mathcal{M}^\triangleright$ and $\mathcal{M}^\triangleleft$, and a solution of the b.v.p. is not a 'saddle connection' as usually defined. Indeed we cannot at this stage assert that $\mathcal{M}^\triangleright$ and $\mathcal{M}^\triangleleft$ are differential manifolds. These issues will be revisited in Part C.

Note that $(\mathcal{M}^\triangleright, \mathcal{U}^\triangleright, \mathcal{B}^\triangleright)$ and $(\mathcal{M}^\triangleleft, \mathcal{U}^\triangleleft, \mathcal{B}^\triangleleft)$ are subsets of $\mathbf{U} \times \Re$ representing sets of starts of solutions of S; they should not be confused with $(\bar{\mathcal{M}}^\triangleright, \bar{\mathcal{U}}^\triangleright, \bar{\mathcal{B}}^\triangleright)$ and $(\bar{\mathcal{M}}^\triangleleft, \bar{\mathcal{U}}^\triangleleft, \bar{\mathcal{B}}^\triangleleft)$ — see (3.24–25) — which are subsets of $\mathbf{U}$ representing starts of solutions of $\bar{S}$. Similarly, the sets defined in (4.2a–d) are subsets of $\mathbf{U}$ defined in terms of systems $S^\wedge$, $S^\vee$, and so forth.

COROLLARY 8. (i) Let (h,θ) be a forward (backward) special solution. Then

$h(z) > 0$ and $\theta(z) < \nu$ eventually as $z \uparrow$ ($z \downarrow$).

If $h(z_\Diamond) > 0$ for some $z_\Diamond$, then $h(z) > 0$ for all $z > z_\Diamond$ ($z < z_\Diamond$).

If $\theta(z_\Diamond) < \nu$ for some $z_\Diamond$, then $\theta(z) \leq \nu$ for all $z > z_\Diamond$ ($z < z_\Diamond$).

(ii) Let (h^*, θ^*) be a star solution. Then

$h^*(z) > 0$ and $\theta^*(z) < \nu$ for all $z \in \Re$.

PROOF. (i) A f.s.s. must satisfy $h(\infty) > 0$ and $\theta(\infty) < \nu$; similarly a b.s.s. must satisfy $h(-\infty) > 0$ and $\theta(-\infty) < \nu$. The assertions then follow from Prop. 8.

(ii) This follows from (i) because a star solution is both a f.s.s. and a b.s.s.$\|$

Prop. 8 and Cor. 8 remain valid (in part trivially) if we allow solutions with $\theta(z) \equiv 0$.

PROPOSITION 9 (Ordering Lemma).

Let $(\pi^i_\Diamond, z_\Diamond) = (h^i_\Diamond, \theta^i_\Diamond, z_\Diamond)$, $i = 0, 1$, be points with $\theta^i_\Diamond > 0$, (not necessarily distinct, but with the same $z_\Diamond$). Let $\phi^i = \phi^i(z) = \phi(z; \pi^i_\Diamond, z_\Diamond)$ with components $h^i(z)$, $\theta^i(z)$ denote the solutions of $S = (F, G)$ through $(\pi^i_\Diamond, z_\Diamond)$, i.e. $\phi^i(z_\Diamond) = \pi^i_\Diamond$.

Similarly, let $\phi^{i\vee}(z) = \phi^\vee(z; \pi^i_\Diamond, z_\Diamond) = (h^{i\vee}(z), \theta^{i\vee}(z))$ denote the solutions of $S^\vee = (F^\vee, G)$ through $(\pi^i_\Diamond, z_\Diamond)$, i.e. $\phi^{i\vee}(z_\Diamond) = \pi^i_\Diamond$; and let $\phi^{i\wedge}(z) = \phi^\wedge(z; \pi^i_\Diamond, z_\Diamond) = (h^{i\wedge}(z), \theta^{i\wedge}(z))$ denote the solutions of $S^\wedge = (F^\wedge, G)$ through $(\pi^i_\Diamond, z_\Diamond)$, i.e. $\phi^{i\wedge}(z_\Diamond) = \pi^i_\Diamond$.

(α) If $h^1_\Diamond \leq h^0_\Diamond$ and $0 < \theta^1_\Diamond \leq \theta^0_\Diamond$, then, on any interval of the form $z_\Diamond < z < \bar{z} < \infty$,

(i) $h^1(z) < h^0(z)$ and $\theta^1(z) < \theta^0(z)$,

(ii) $h^1(z) < h^{0\vee}(z)$ and $\theta^1(z) < \theta^{0\vee}(z)$,

(iii) $h^0(z) > h^{1\wedge}(z)$ and $\theta^0(z) > \theta^{1\wedge}(z)$,

provided that the following hold: in each line both solutions exist on $[z_\Diamond, \bar{z}]$; for at least one of the solutions, the h-coordinate remains positive on this interval in case (i), non-negative in cases (ii) and (iii); and, in case (i), the points $\pi^i_\Diamond$ are distinct.

If ρ and z^ρ are chosen as in (3.51) and $z_\Diamond \geq z^\rho$, then $\phi^{1\wedge}$, $\phi^{0\vee}$ may be replaced throughout by $\phi^{1\wedge\rho}$, $\phi^{0\vee\rho}$, defined respectively as the solutions of $S^{\wedge\rho}$, $S^{\vee\rho}$ through $(\pi^\wedge_\Diamond, z_\Diamond)$ and $(\pi^0_\Diamond, z_\Diamond)$ for $z \geq z_\Diamond$.

(β) If $h^1_\Diamond \leq h^0_\Diamond$ and $0 < \theta^0_\Diamond \leq \theta^1_\Diamond$, then, on any interval of the form $z_\Diamond > z > \bar{z} > -\infty$,

(i) $h^1(z) < h^0(z)$ and $\theta^1(z) > \theta^0(z)$,

(ii) $h^0(z) > h^{1\vee}(z)$ and $\theta^0(z) < \theta^{1\vee}(z)$,

(iii) $h^1(z) < h^{0\wedge}(z)$ and $\theta^1(z) > \theta^{0\wedge}(z)$,

provided that the following hold: in each line both solutions exist on $[\bar{z}, z_\Diamond]$; for at least one of the solutions, the h-co-ordinate remains positive on this interval in case (i), non-negative in cases (ii) and (iii); and, in case (i), the points $\pi^i_\Diamond$ are distinct.

If δ and z^δ are chosen as in (3.43) and $z_\Diamond \leq z^\delta$, then $\phi^{1\wedge}$, $\phi^{0\vee}$ may be replaced throughout by $\phi^{1\wedge\delta}$, $\phi^{0\vee\delta}$, defined respectively as solutions of $S^{\wedge\delta}$, $S^{\vee\delta}$ through $(\pi^\wedge_\Diamond, z_\Diamond)$ and $(\pi^0_\Diamond, z_\Diamond)$ for $z \leq z_\Diamond$.

Proof (α)(i). Suppose first that $h^0_\Diamond > h^1_\Diamond > 0$ and $\theta^0_\Diamond > \theta^1_\Diamond > 0$. Then $h^0(z) > h^1(z)$ for z in a right neighbourhood of $z_\Diamond$, and since $(d/dz)(\ln\theta^0 - \ln\theta^1) = h^0 - h^1$ it follows that $\theta^0(z) > \theta^1(z)$ in this neighbourhood; moreover, the latter inequality persists as long as $h^0(z) > h^1(z)$ for increasing z. If some $\tilde{z} \in (z_\Diamond, \bar{z})$ were reached with $h = h^0(\tilde{z}) = h^1(\tilde{z}) > 0$, then (for variables evaluated at $\tilde{z}$) we should have $(d/dz)(h^0 - h^1) = F(h, \theta^0, \tilde{z}) - F(h, \theta^1, \tilde{z}) = (2/\sigma^2)h(\theta^0 - \theta^1) > 0$, so that in fact the inequality $h^0 > h^1$ would persist, contrary to assumption.

Now suppose that $h^0_\Diamond = h^1_\Diamond > 0$ and $\theta^0_\Diamond > \theta^1_\Diamond > 0$. Then $(d/dz)(h^0 - h^1) > 0$ for z to the right of $z_\Diamond$ and the rest of the argument proceeds as before. Alternatively, suppose that $h^0_\Diamond > 0$, $h^1_\Diamond < h^0_\Diamond$ and $\theta^1_\Diamond = \theta^1_\Diamond > 0$. Then $(d/dz)(\ln\theta^0 - \ln\theta^1) = (h^1 - h^0)$, hence $\theta^1 < \theta^0$ for z in a right neighbourhood of $z_\Diamond$, and again the argument proceeds as before.$\|$

(α)(ii) Suppose initially that $h^0_\Diamond > h^1_\Diamond > 0$ and $\theta^0_\Diamond > \theta^1_\Diamond > 0$. Then it follows as before that $h^{0\vee}(z) > h^1(z)$ for z to the right of $z_\Diamond$, and as long as this inequality persists it follows that $\theta^{0\vee}(z) > \theta^1(z)$ also. If some $\tilde{z} \in (z_\Diamond, \tilde{z})$ were reached with $h = h^{0\vee}(\tilde{z}) = h^1(\tilde{z}) > 0$, then we should have, at $\tilde{z}$,

$$(4.3) \quad (d/dz)(h^{0\vee} - h^1) = F^\vee(h, \theta^{0\vee}, \tilde{z}) - F(h, \theta^1, \tilde{z}) > F(h, \theta^{0\vee}, \tilde{z}) - F(h, \theta^1, \tilde{z}) > 0,$$

because $F^{\vee} > F$ and $\partial F/\partial\theta = (2/\sigma^2)h > 0$, so that once again the inequality $h^{0\vee} > h^1$ would persist. This last assertion remains true even if $h = h^{0\vee}(\tilde{z}) = h^1(\tilde{z}) = 0$, since only the second strict inequality in (3) need be replaced by a weak one. In case either $(h^0_\diamondsuit = h^1_\diamondsuit > 0, \theta^0_\diamondsuit > \theta^1_\diamondsuit)$ or $(h^0_\diamondsuit > 0, h^1_\diamondsuit < h^0_\diamondsuit, \theta^1_\diamondsuit = \theta^0_\diamondsuit)$, the argument is modified as under (α)(i).‖

The proofs of the remaining assertions under (α), and those under (β), are analogous.

REMARKS: (1) The requirement that one of the h-coordinates in each line remain positive (or non-negative in cases (ii) and (iii)) seems to be essential.

However, the inequality $0 < \theta^1_\diamondsuit \leq \theta^0_\diamondsuit$ in Part (α) may be replaced by $0 \leq \theta^1_\diamondsuit < \theta^0_\diamondsuit$, similarly $0 < \theta^0_\diamondsuit \leq \theta^1_\diamondsuit$ in Part (β) may be replaced by $0 \leq \theta^0_\diamondsuit < \theta^1_\diamondsuit$. If in either Part, the inequality in question is replaced by $0 = \theta^1_\diamondsuit = \theta^0_\diamondsuit$, it can be shown that the inequalities relating to the h-co-ordinate remain valid.

(2) Part (α) of the Lemma is essentially a version of a theorem of Kamke [1932] T.6 or [1943] A.23.2 on what are now called 'co-operative' systems, see Hirsch [1984], Smith [1988] and [1995] for surveys. Briefly, the system $S = (F, G)$ is 'co-operative' at z if the off-diagonal elements of the Jacobian matrix are positive, which here means that $\partial F/\partial\theta = (2/\sigma^2)h > 0$ and $\partial G/\partial h = \theta > 0$. It is however more efficient to give a direct proof for the present model.

In general, the inequalities (α)(i-iii) in Prop. 9 cannot be extended to limits as $z \to \infty$ even if both solutions in question are defined on $[z_\diamondsuit, \infty)$, nor can (β)(i-iii) be extended to limits as $z \to -\infty$. Rather than set out comparisons between *limits* at this stage, it is convenient to state some comparisons between *starts* of solutions converging to saddle points of asymptotic systems:

PROPOSITION 10 (Uniqueness Lemma for Special Starts).

Let $(\pi^i_\diamondsuit, z_\diamondsuit) = (h^i_\diamondsuit, \theta^i_\diamondsuit, z_\diamondsuit)$, $i = 0, 1$, be points with $\theta^i_\diamondsuit > 0$, and either $h^0_\diamondsuit \geq 0$ or $h^1_\diamondsuit \geq 0$ (or both). Let $\phi^i(z) = (h^i(z), \theta^i(z)) = \phi^i(z; \pi^i_\diamondsuit, z)$ denote the solutions of $S = (F, G)$ through $(\pi^i_\diamondsuit, z_\diamondsuit)$. Similarly, let $\phi^{i\vee}(z) = (h^{i\vee}, \theta^{i\vee}) = \phi^{\vee}(z; \pi^i_\diamondsuit, z_\diamondsuit)$ denote the solutions of $S^{\vee} = (F^{\vee}, G)$ through $(\pi^i_\diamondsuit, z_\diamondsuit)$.

(α)(i) If $\phi^0(z)$ and $\phi^1(z)$ are defined on $[z_\diamondsuit, \infty)$ and both solutions converge as $z \to \infty$ to the saddle point π^*_∞ of S_∞, then *either* $\pi^0_\diamondsuit = \pi^1_\diamondsuit$, *or* $(h^0_\diamondsuit \neq h^1_\diamondsuit, \theta^0_\diamondsuit \neq \theta^1_\diamondsuit)$ and

$$(\theta^0_\diamondsuit - \theta^1_\diamondsuit)(h^0_\diamondsuit - h^1_\diamondsuit) < 0.$$

(α)(ii) If $b \geq 1$ and both $\phi^1(z)$ and $\phi^{0\vee}(z)$ are defined on $[z_\Diamond, \infty)$ and both converge as $z \to \infty$ to the saddle point π^*_∞ of S_∞ (which for $b \geq 1$ is also the saddle point of $S^\vee$), then $(h^0_\Diamond \neq h^1_\Diamond, \theta^0_\Diamond \neq \theta^1_\Diamond)$ and

$$(\theta^{0\vee}_\Diamond - \theta^1_\Diamond)(h^{0\vee}_\Diamond - h^1_\Diamond) < 0.$$

(β)(i) If $\phi^0(z)$ and $\phi^1(z)$ are defined on $(-\infty, z_\Diamond]$ and both converge as $z \to -\infty$ to the saddle point $\pi^*_{-\infty}$ of $S_{-\infty}$, then *either* $\pi^0_\Diamond = \pi^1_\Diamond$, *or* $(h^0_\Diamond \neq h^1_\Diamond, \theta^0_\Diamond \neq \theta^1_\Diamond)$ and

$$(\theta^0_\Diamond - \theta^1_\Diamond)(h^0_\Diamond - h^1_\Diamond) > 0.$$

(β)(ii) If $b \leq 1$ and both $\phi^1(z)$ and $\phi^{0\vee}(z)$ are defined on $(-\infty, z_\Diamond]$ and both converge as $z \to -\infty$ to the saddle point $\pi^*_{-\infty}$ of $S_{-\infty}$ (which for $b \leq 1$ is also the saddle point of $S^\vee$), then $(h^0_\Diamond \neq h^1_\Diamond, \theta^0_\Diamond \neq \theta^1_\Diamond)$ and

$$(\theta^{0\vee}_\Diamond - \theta^1_\Diamond)(h^{0\vee}_\Diamond - h^1_\Diamond) > 0.$$

Proof. (α)(i) Suppose that $h^0_\Diamond \neq h^1_\Diamond$, $\theta^0_\Diamond \neq \theta^1_\Diamond$ and $h^0_\Diamond > 0$. We assume that the inequality $(\theta^0_\Diamond - \theta^1_\Diamond)(h^0_\Diamond - h^1_\Diamond) < 0$, is false, say that $h^1_\Diamond \leq h^0_\Diamond$, $0 < \theta^1_\Diamond \leq \theta^0_\Diamond$, and derive a contradiction. By Prop. 8(i), we have $h^0(z) > 0$ on $(z_\Diamond, \infty)$, and then, according to Prop. 9(α)(i), the inequalities $h^1(z) < h^0(z)$, $\theta^1(z) < \theta^0(z)$ hold on $(z_\Diamond, \infty)$. Since $h^1(z)$ and $h^0(z)$ go to the same positive limit, both are positive for large z, say for $z \geq \bar{z} \geq z_\Diamond$, so that $0 < h^1(z) < h^0(z)$ and $0 < \theta^1(z) < \theta^0(z)$ for $z \in [z_\Diamond, \infty)$. In particular, $(h^1(\bar{z}), \theta^1(\bar{z}), \bar{z})$ and $(h^0(\bar{z}), \theta^0(\bar{z}), \bar{z})$ are again forward starts of solutions converging to π^*_∞ for which $(\theta^0(\bar{z}) - \theta^1(\bar{z}))(h^0(\bar{z}) - h^1(\bar{z})) > 0$, so that it will be sufficient to show that these conditions lead to contradiction.

Suppose first that S_∞ is Type 1, $n > 0$, and both $\phi^0(z)$ and $\phi^1(z)$ converge to $\pi^*_\infty = (1, n)$. Then $\theta^0(z)/\theta^1(z) \to 1$, $\ln[\theta^0(z)/\theta^1(z)] \to 0$. On the other hand, $\theta^0(\bar{z})/\theta^1(\bar{z}) > 1$, and since $d \ln \theta/dz = h - 1$ it follows from $h^0(z) > h^1(z)$ that $\ln \theta^0(z) - \ln \theta^1(z)$ increases on $[\bar{z}, \infty)$, leading to contradiction.

Alternatively, suppose that S_∞ is Type 0, $q > 0 \geq n$, and that both solutions converge to $\pi^*_\infty = (h^+_\infty, 0)$. For $z > \bar{z}$, an application of the mean value theorem to the difference $F^0(z) - F^1(z)$, where $F^i(z) = F(h^i(z), \theta^i(z), z)$, gives, in abridged notation,

$$(d/dz)(h^0 - h^1) = F^0 - F^1 = (h^0 - h^1)F^\eta_h + (\theta^0 - \theta^1)F^\eta_\theta; \tag{4.4a}$$

here $F_h = \partial F/\partial h$, $F_\theta = \partial F/\partial\theta$, and the superscript η indicates that the derivatives are evaluated at some point

(4.4b)

$(h^\eta, \theta^\eta) = ((1-\eta)h^1 + \eta h^0), (1-\eta)\theta^1 + \eta\theta^0)),\ 0 < \eta = \eta(z) < 1,\ h^i = h^i(z),\ \theta^i = \theta^i(z).$

Now $F^\eta_\theta = (2/\sigma^2)h^\eta > 0$, and

$$F^\eta_h = 2bh^\eta + (2/\sigma^2)[\theta^\eta - Q - A(z)] \to (2/\sigma^2)[b\sigma^2 h^+_\infty - Q] = (2/\sigma^2)[Q^2 + 2m\sigma^2]^{\frac{1}{2}} > 0$$

as $z \to \infty$ by the definition of h^+_∞, see (3.7), (3.10) and fn.6 of Section 3. On dividing (4a) by $h^0(z) - h^1(z)$ and recalling that $\theta^0 - \theta^1 > 0$ and $h^0 - h^1 > 0$ for $\bar{z} < z < \infty$, we obtain

$$(d/dz)\ln(h^0 - h^1) > F^\eta_h \to (2/\sigma^2)[b\sigma^2 h^+_\infty - Q] > 0, \quad z \to \infty.$$

Thus $h^0 - h^1$ is positive and increasing for large z, contrary to the assumption that $h^0 - h^1 \to 0$, again a contradiction.

There remains, a priori, the possibility that $\pi^0_\diamondsuit \neq \pi^1_\diamondsuit$ but $(\theta^0_\diamondsuit - \theta^1_\diamondsuit)(h^0_\diamondsuit - h^1_\diamondsuit) = 0$ because either $(\theta^1_\diamondsuit = \theta^0_\diamondsuit, h^1_\diamondsuit \neq h^0_\diamondsuit, h^0_\diamondsuit > 0)$ or $(\theta^1_\diamondsuit \neq \theta^0_\diamondsuit, h^1_\diamondsuit = h^0_\diamondsuit > 0)$. In these cases it again follows from Props.8(i) and 9(α)(i) that $0 < h^1(z) < h^0(z)$ and $0 < \theta^1(z) < \theta^0(z)$ for $z \geq \bar{z}$ large enough, leading to contradiction as above.$\|$

The implication of these results is that distinct points $(\pi^0_\diamondsuit, z_\diamondsuit)$ and $(\pi^1_\diamondsuit, z_\diamondsuit)$ with $h^0_\diamondsuit \vee h^1_\diamondsuit > 0$ and $(\theta^0_\diamondsuit - \theta^1_\diamondsuit)(h^0_\diamondsuit - h^1_\diamondsuit) \geq 0$ cannot define starts of forward solutions $\phi^0 = \phi(z; \pi^0_\diamondsuit, z_\diamondsuit)$ and $\phi^1 = \phi(z; \pi^1_\diamondsuit, z_\diamondsuit)$, both of which converge to π^*_∞.

(α) (ii) The proof is similar to that for Prop. 10(α)(i) but with ϕ^0 replaced by $\phi^{0\vee}$. We again assume that $(h^0_\diamondsuit \neq h^1_0, \theta^0_\diamondsuit \neq \theta^1_\diamondsuit)$, $h^0_\diamondsuit > 0$ and $(\theta^0_\diamondsuit - \theta^1_\diamondsuit)(h^0_\diamondsuit - h^1_\diamondsuit) > 0$, say that $(h^1_\diamondsuit < h^0_\diamondsuit, \theta^1_\diamondsuit < \theta^0_\diamondsuit)$, and obtain a contradiction. Using Props. 8(i) and 9(α)(ii), it is found that $0 < h^1(z) < h^{0\vee}(z)$ and $0 < \theta^1(z) < \theta^{0\vee}(z)$ for z large enough, say for $z \leq \bar{z} \leq z_\diamondsuit$. Now $(h^1(\bar{z}), \theta^1(\bar{z}), \bar{z})$ and $(h^{0\vee}(\bar{z}), \theta^{0\vee}(\bar{z}), \bar{z})$ are both starts of forward solutions converging to π^*_∞ for which $(\theta^{0\vee}(\bar{z}) - \theta^1(\bar{z}))(h^{0\vee}(\bar{z}) - h^1(\bar{z})) > 0$. Also, $h^0_\infty > 1/b$ for S_∞ of either Type, so $\bar{z}$ may be chosen large enough so that $h^1(z) > 1/b$ and $h^{0\vee} > 1/b$ for $z \geq \bar{z}$.

In case S_∞ is Type 1, the argument that $\phi^{0\vee}(z)$ and $\phi^1(z)$ cannot both converge to $\pi^*_\infty = (1, n)$ with $n > 0$ proceeds as for 10(α)(i). In case S_∞ is Type 0, so that $q > 0 \geq n$, with $\pi^*_\infty = (h^+_\infty, 0)$, recall that in the region $\{h \geq 1/b\}$ we have $F^\vee = F_\infty$, see (3.40a), and that

$$F(h,\theta,z) = F_\infty(h,\theta) + (2/\sigma^2)[M/b - hA], \quad \text{see (0.1) and (3.1).}$$

Writing $F^1_\infty(z) = F_\infty(h^1(z), \theta^1(z), z)$, and $F^{0\vee}_\infty(z) = F_\infty(h^{0\vee}(z), \theta^{0\vee}(z))$, the first equation in (4a) is replaced by

$$(d/dz)(h^{0\vee} - h^1) = F^{0\vee}_\infty - F^1_\infty - (2/\sigma^2)[M/b - h^1 A].$$

The term $-(2/\sigma^2)[M/b - h^1 A]$ is positive for $z \geq \bar{z}$ since $A > M$ and $h^1 > 1/b$, and so can be dropped in the rest of the proof, which proceeds as before with F_∞ in place of F.‖

Reviewing the final paragraph of the proof of Prop. 10(α)(i), note that in the present situation, $\pi^0_\Diamond$ and $\pi^1_\Diamond$ are necessarily distinct because $F^{0\vee} \neq F^1$. However, using Prop. 8(i) and 9(α)(ii) as above, the conditions that *either* $(\theta^1_\Diamond = \theta^0_\Diamond, h^1_\Diamond \neq h^0_\Diamond, h^0_\Diamond > 0)$ *or* $(\theta^1_\Diamond \neq \theta^0_\Diamond, h^1_\Diamond = h^0_\Diamond)$ — so that $(\theta^0_\Diamond - \theta^1_\Diamond, h^0_\Diamond - h^1_\Diamond) = 0$, again lead to contradiction.‖

The proofs of 10(β)(i) and 10(β)(ii) are analogous to those of 10(α)(i) and 10(α)(ii) allowing for changes of direction and sign.

COROLLARY 10.1. For given $z_\Diamond$ and $\theta_\Diamond > 0$, there is at most one $h_\Diamond > 0$ defining a start of a f.s.s., and if there is one such $h_\Diamond$ then it is the only one of either sign. If for given $z_\Diamond$, there is a θ-interval $I = (\theta^0, \theta^1)$ with $\theta^0 > 0$ such that, for each $\theta \in I$, there is $h = h(\theta, z_\Diamond) > 0$ for which $(h, \theta, z_\Diamond)$ is a f.s. start, then the points $(h, \theta, z_\Diamond)$ define the graph of a function $h = f(\theta, z_\Diamond)$ on I which is positive and *decreasing* in θ. Again, there is for given $z_\Diamond$ and $h_\Diamond > 0$ at most one $\theta_\Diamond > 0$ defining a start of a f.s.s. Similarly for b.s.s. in place of f.s.s., with $f(\theta, z_\Diamond)$ replaced by a function $g(\theta, z_\Diamond)$ which is positive and *increasing* in θ.

These results follow immediately from Prop. 10(α)(i) and (β)(i).

COROLLARY 10.2. (α)(i) For $b > 1$, a path of S which passes through a point of $[\![\mathcal{B}^{\vee\triangleright}]\!] = \mathcal{B}^{\vee\triangleright} \cup \mathcal{M}^{\vee\triangleright} = \{(h,\theta)\colon h \leq f^\vee(\theta), \theta > 0\}$ cannot converge to π^*_∞ as $z \to \infty$.

(α)(ii) For $b \leq 1$ and ρ, z^ρ chosen as in (3.51), a path of S which passes through a point of $[\![\mathcal{B}^{\vee\rho}]\!] = \mathcal{B}^{\vee\rho} \cup \mathcal{M}^{\vee\rho} = \{(h,\theta)\colon h \leq f^{\vee\rho}\}$ at some $z_\Diamond \geq z^\rho$ cannot converge to π^*_∞ as $z \to \infty$.

(β)(i) For $b \leq 1$, a path of S which passes through a point of $[\![\mathcal{B}^{\vee\triangleleft}]\!] = \mathcal{B}^{\vee\triangleleft} \cup \mathcal{M}^{\vee\triangleleft} = \{(h,\theta)\colon h \geq g^{\vee}(\theta), \theta > 0\}$ cannot converge to $\pi^*_{-\infty}$ as $z \to -\infty$.

(β)(ii) For $b \geq 1$ and δ, z^{δ} chosen as in (3.43), a path of S which passes through a point of $[\![\mathcal{B}^{\vee\delta}]\!] = \mathcal{B}^{\vee\delta} \cup \mathcal{M}^{\vee\delta} = \{h \geq g^{\vee\delta}(\theta), \theta > 0\}$ at some $z_{\Diamond} \leq z^{\delta}$ cannot converge to $\pi^*_{-\infty}$ as $z \to -\infty$.

PROOF of Cor 10.2(α)(i). Let $\phi^1(z) = \phi(z;\pi^1_{\Diamond}, z_{\Diamond})$ again be a solution of S which at some $z_{\Diamond}$ passes through a point $\pi^1_{\Diamond} = (h^1_{\Diamond}, \theta^1_{\Diamond}) \in [\![\mathcal{B}^{\vee\triangleright}]\!] = \{(h,\theta)\colon h \leq f^{\vee}(\theta), \theta > 0\}$. Then $\phi^1(z_{\Diamond}) = \pi^1$ and $h^1_{\Diamond} \leq f^{\vee}(\theta^1_{\Diamond})$. We may assume that $\phi^1(z)$ is defined for $z \in [z_{\Diamond}, \infty)$. Bearing in mind that paths of S can enter $\mathcal{B}^{\vee\triangleright}$ via the boundary $\{h = f^{\vee}(\theta)\}$ but not escape, cf.(3.41), it follows that the inequality $h^1(z) \leq f^{\vee}[\theta^1(z)]$ is preserved for $z \geq z_{\Diamond}$. Now consider the forward solution $\phi^{0\vee}$ of $S^{\vee}$ defined by setting $\phi^{0\vee} = \phi^{\vee}(z;\pi^0_{\Diamond}, z_{\Diamond})$ with $\pi^0_{\Diamond} = (h^0_{\Diamond}, \theta^0_{\Diamond}) = (f^{\vee}(\theta^1_{\Diamond}), \theta^1_{\Diamond})$. This solution *does* converge to π^*_{∞} as $z \to \infty$ and its path is contained in $\mathcal{M}^{\vee\triangleright} = \{h = f^{\vee}(\theta)\}$. Now $h^1(z) \leq f^{\vee}[\theta^1(z)] = h^{0\vee}(z)$, and since $F < F^{\vee}$, see (3.40), the difference $h^{0\vee}(z) - h^{\vee}(z)$ increases as $z \uparrow$ and so becomes strictly positive. It follows that $\phi^1(z) = (h^1(z), \theta^1(z))$ does not converge to π^*_{∞}.‖ The proof of (β)(i) is analogous, with $f^{\vee}$ replaced by $g^{\vee}$, $h^1 \leq f^{\vee}(\theta^1)$ by $h^1 \geq g^{\vee}(\theta^1)$ etc.

As to (α)(ii), note that $(1,n)$ is exterior to $\mathcal{B}^{\vee\rho}$, so that the assertion follows directly from the fact that a path once in $[\![\mathcal{B}^{\vee\rho}]\!]$ at some $z_{\Diamond} \geq z^{\rho}$ cannot leave that set as $z \uparrow$; in this case Prop. 10 need not be invoked. Similarly for (β)(ii).‖

PROPOSITION 11 (Convergence Lemma).

Every solution of $S = (F, G)$ which is defined and bounded on a forward (backward) unbounded interval converges to a finite limit as $z \to \infty$ ($z \to -\infty$).

More precisely, for solutions $\phi(z) = (h(z), \theta(z))$ with $\theta(z) > 0$, we have:

(α)(i) For $b \geq 1$, a solution defined on an interval $[z_{\Diamond}, z_+)$ becomes unbounded as $z \uparrow$ iff its path ever enters $\mathcal{U}^{\wedge\triangleright}$. Otherwise it converges as $z \to z_+ = \infty$ to the stable node $(h^-_{\infty}, 0)$ of S_{∞} iff its path is ultimately in $\mathcal{B}^{\vee\triangleright}$, and to the saddle point of S_{∞} iff its path lies entirely in $\mathcal{C}^{\triangleright}$.

(α)(ii) For $b \leq 1$ and ρ, z^{ρ} chosen as in (3.51), a solution defined on an interval $[z_{\Diamond}, z_+)$, where $z_{\Diamond} \geq z^{\rho}$, becomes unbounded as $z \uparrow$ iff its path ever enters $\mathcal{U}^{\wedge\rho}$. Otherwise it converges as $z \to z_+ = \infty$ to the stable node of S_{∞} iff its path is ultimately in $\mathcal{B}^{\vee\rho}$, and to the saddle point of S_{∞} iff its path remains in $\mathcal{C}^{\rho}$.

(β)(i) For $b \leq 1$, a solution defined on an interval $(z_-, z_\Diamond]$ becomes unbounded as $z \downarrow$ iff its path ever enters $\mathcal{U}^{\wedge\triangleleft}$. Otherwise it converges as $z \to z_- = -\infty$ to the unstable node $(h^+_{-\infty}, 0)$ of $S_{-\infty}$ iff its path is ultimately in $\mathcal{B}^{\vee\triangleleft}$, and to the saddle point of $S_{-\infty}$ iff its path lies entirely in $\mathcal{C}^{\triangleleft}$.

(β)(ii) For $b \geq 1$, and δ, z^δ chosen as in (3.43), a solution defined on an interval $(z_-, z_\Diamond]$, where $z_\Diamond \leq z^\delta$, becomes unbounded as $z \downarrow$ iff its path ever enters $\mathcal{U}^{\wedge\delta}$. Otherwise it converges as $z \to z_- = -\infty$ to the unstable node of $S_{-\infty}$ iff its path is ultimately in $\mathcal{B}^{\vee\delta}$, and to the saddle point of $S_{-\infty}$ iff its path remains in $\mathcal{C}^{\delta}$.

Recall that by Props. 1 and 2, if a solution of S stays bounded as $z \uparrow$ ($z \downarrow$), then $z_+ = \infty$ ($z_- = -\infty$), and if the solution converges to a point then that point is a stationary point of S_∞ ($S_{-\infty}$). Thus the problem addressed by the present Proposition is to prove convergence of bounded solutions and to assign solutions to the available limit points.

Proof. Let $\phi = \phi(z) = (h(z), \theta(z) \colon z_- < z < z_+)$ be an arbitrary solution of S with $\theta(z) > 0$ and let $(\pi_\Diamond, z_\Diamond)$ with $\pi_\Diamond = (h_\Diamond, \theta_\Diamond)$ be a point through which the solution passes.

(α)(i) *Forward Motion*, $b \geq 1$. Referring to (4.2 a–b), we note that the sets $\mathcal{U}^{\wedge\triangleright}$, $\mathcal{C}^{\triangleright}$, $\mathcal{B}^{\wedge\triangleright}$ defined there partition the half-plane $\{\theta > 0)\}$. Of these sets, $\mathcal{U}^{\wedge\triangleright}$ is an unbounded open region from which paths of S and S_∞ cannot escape as $z \uparrow$ and (as shown earlier) all paths of S_∞ which enter this region become unbounded. According to Prop. 2(iv), the same is true of paths of S. Any path which reaches $\mathcal{M}^{\wedge\triangleright}$ passes immediately into $\mathcal{U}^{\wedge\triangleright}$. Also, any path which reaches $\mathcal{M}^{\vee\triangleright}$ passes immediately into $\mathcal{B}^{\vee\triangleright}$ and cannot escape as $z \uparrow$. Thus a path is ultimately in one of $\mathcal{U}^{\wedge\triangleright}$, $\mathcal{B}^{\vee\triangleright}$ or $\mathcal{C}^{\triangleright}$.

If a given solution ϕ never enters $\mathcal{U}^{\wedge\triangleright}$, it is bounded for $z \geq z^\rho$. Indeed, the path is bounded above and to the right by the graph of $f^\wedge$; and if $h(z)$ assumes negative values these are bounded because of the term bh^2 in F and the boundedness of $\theta(z)$ — see Figs. 3–4. Thus $z_+ = \infty$, and the forward limit set $\Pi^\triangleright$ is not empty and is contained in $\{(h, \theta) \colon h \leq f^\wedge(\theta), \theta \geq 0\}$. We wish to characterise this set.

There are three possibilities a priori concerning the ultimate path behaviour:

(I) $\theta(z)$ is non-decreasing and $h(z) \geq 1$ for all z large enough, say for $z \geq z_\Diamond$.

(II) $\theta(z)$ is non-increasing and $h(z) \leq 1$ for all z large enough, say for $z \geq z_\Diamond$.

(III) $\theta(z)$ is not (weakly) monotonic and $h(z)-1$ does not have constant definite sign for large z.

In case (I), $\theta(z)\uparrow$ some θ_∞, and since the path cannot terminate in the interior of $\{h\geq 1\}$, and by assumption does not pass into $\mathcal{U}^{\wedge\triangleright}$ or into $\{h<1\}$, we must have $\theta_\infty\leq\nu$ and $h(z)\to 1$; but then $(1,\theta_\infty)$ must be a stationary point of S_∞ by Prop. 2(ii), which is possible only if $\theta_\infty = n\geq 0$. If $n<0$, case (I) cannot occur for bounded solutions.

In case (II), $\theta(z)\downarrow$ some θ_∞, and since the path cannot terminate in the interior of $\{h\leq 1\}$ and does not leave this set we have either a limit $(1,n)$ with $n>0$, or $\theta_\infty=0$. Consider the latter case. Any point of the forward limit set $\Pi^\triangleright$ must be of the form $(\bar{h},0)$ with $\bar{h}\leq 1$, and for such a point there must be a sequence $z_k\uparrow\infty$ with $h(z_k)\to\bar{h}$. Consequently, if there are two points in $\Pi^\triangleright$, say with $\bar{h}=h_\alpha$ and $\bar{h}=h_\beta$, $h_\alpha<h_\beta$, then all $\bar{h}$ in the interval $[h_\alpha,h_\beta]$ must also define points $(\bar{h},\theta_\infty)$ of $\Pi^\triangleright$; thus $\Pi^\triangleright$ has the form $I\times\{0\}$, where I is an interval bounded above by 1. On the other hand, $\Pi^\triangleright$ must be the union of complete paths of S_∞ by Prop. 2(ii). Now, the paths of S_∞ lying on the vertical axis are as follows: the stationary points $(h^+_\infty,0)$ and $(h^-_\infty,0)$, and the intervals $I^-_\infty=\{(h,0)\colon h<h^-_\infty\}$, $I^0_\infty=\{(h,0)\colon h^-_\infty<h<h^+_\infty\}$ and $I^+_\infty=\{(h,0)\colon h\geq h^+_\infty\}$, cf. Section 3, fn.11. It is impossible for the whole of I^-_∞ to define limit points because $h(z)$ is bounded below. Also, I^+_∞ may be left aside since it is not bounded above by 1. Further, if $n>0$, then $h^+_\infty>1$, so $(h^+_\infty,0)$ and I^0_∞ do not satisfy the stated bound; thus in this case $\Pi^\triangleright$ consists of the single point $(h^-_\infty,0)$, the stable node of S_∞, and this point is the limit. Now suppose $n\leq 0$, so that $h^+_\infty\leq 1$. It is impossible for every point of I^0_∞ to be a limit point of the solution because $F_\infty(h,0)<0$ for $h\in(h^-_\infty,h^+_\infty)$, and since $F\to F_\infty$ uniformly on (h,θ)-compacts it follows that, for $\epsilon>0$ small enough, there is $z(\epsilon)$ such that $F(h(z),\theta(z),z)<-\epsilon$ for $h\in(h^-_\infty+\epsilon,h^+_\infty-\epsilon)$ and $z>z(\epsilon)$, so that eventually the interval $(h^-_\infty+\epsilon,h^+_\infty-\epsilon)$ can be traversed only in the downward direction. Thus, if $n\leq 0$ the only possible limit points are $(h^-_\infty,0)$ and $(h^+_\infty,0)$, the stable node and saddle point of S_∞, and one of these must be the limit.

Consider now case (III). There must be a sequence (z_k) such that, for each $k=1,2,\ldots$, we have $h(z)-1\geq 0$ on (z_{2k-1},z_{2k}), $h(z)-1\leq 0$ on (z_{2k},z_{2k+1}), and moreover $h(z)-1$ does not vanish identically on any of these intervals or on any neighbourhood of the z_k. Note that $\theta(z)$ is non-decreasing on each (z_{2k-1},z_{2k}), non-increasing on each (z_{2k},z_{2k+1}). Writing $F(z)=F(h(z),\theta(z),z)$ and $\eta(z)=A(z)-M(z)/b$, it follows from (0.1) that if $h(z)=1$ then $F(z)$ has the same sign as $\theta(z)-n-\eta(z)$. Now

$F(z_{2k-1}) \geq 0 \geq F(z_{2k})$, and since $\eta(z) \to 0$ as $z \to \infty$ we have $\theta(z_{2k-1}) \to n$ and $\theta(z_{2k}) \to n$. It then follows from the monotonicity of $\theta(z)$ on each interval that $\theta(z) \to n$ as $z \to \infty$. Consequently the forward limit set must have the form $\{(h, n) \colon h \in I\}$ where I is an interval, and this set must be a union of complete paths of S_∞. If $n > 0$, this is possible only if the set reduces to the singleton $(1, n)$. If $n = 0$, an argument like that given in case (II) yields the same conclusion; (in fact we must have $h^+_\infty = 1$). If $n < 0$, case (III) cannot occur.

So far we have shown that, for $b > 1$, a path of S becomes unbounded if it is ever in $\mathcal{U}^{\triangleright}$, otherwise it is in $\mathcal{B}^{\vee\triangleright}$ or $\mathcal{C}^{\triangleright}$ for large z and converges either to the stable node or to the saddle point of S_∞. If a path is ever in $\mathcal{B}^{\vee\triangleright}$, then according to Cor.10.2(α)(i), it cannot converge to the saddle point of S_∞ and so converges to the stable node of S_∞. Also, as noted earlier — see (3.41) — paths may leave but not enter $\mathcal{C}^{\triangleright}$ as $z \uparrow$, so that if a path converges to the saddle point of S_∞ it must be in $\mathcal{C}^{\triangleright}$ for all $z \geq z_\Diamond$. Obviously each of the three occurrences of 'if' in this paragraph may be replaced by 'iff'. This completes the proof of Prop. 11(α)(i).‖

(α)(ii) *Forward Motion*, $b \leq 1$. A proof like that under (α)(i) works if we restrict z to an interval $[z_\Diamond, z_+)$ with $z_\Diamond \geq z^\rho$ and consider functions $f^{\wedge\rho}$, $f^{\vee\rho}$ and sets $\mathcal{U}^{\wedge\rho}$, $\mathcal{B}^{\vee\rho}$, $\mathcal{C}^\rho$ in place of $f^\wedge$, $f^\vee$, $\mathcal{U}^{\wedge\triangleright}$, $\mathcal{B}^{\vee\triangleright}$, $\mathcal{C}^{\triangleright}$. Note that $b \leq 1$ implies $n > 0$ by (0.9–10), so that only Type 1 systems S_∞, $S^{\wedge\rho}$, $S^{\vee\rho}$ need be considered, see (3.51)ff, and the saddle point of S_∞ is $(1, n)$ while the stable node is $(h^-_\infty, 0)$.

(β)(i) *Backward Motion*, $b \leq 1$. The argument is similar to that under (α)(i) with $z \downarrow z_-$ replacing $z \uparrow z_+$, $g^\wedge$, $g^\vee$, $\mathcal{U}^{\wedge\triangleleft}$, $\mathcal{B}^{\vee\triangleleft}$, $\mathcal{C}^{\triangleleft}$ replacing $f^\wedge$, $f^\vee$, $\mathcal{U}^{\wedge\triangleright}$, $\mathcal{B}^{\vee\triangleright}$, $\mathcal{C}^{\triangleright}$ and the limits $(1, N)$, $(h^-_{-\infty}, 0)$, $(h^+_{-\infty}, 0)$ replacing $(1, n)$, $(h^+_\infty, 0)$, $(h^-_\infty, 0)$.

(β)(ii) *Backward Motion*, $b \geq 1$. The argument is similar to that under (β)(i) if we restrict z to an interval $(z_-, z_\Diamond)$ with $z_\Diamond \leq z^\delta$ and consider functions $g^{\wedge\delta}$, $g^{\vee\delta}$ and sets $\mathcal{U}^{\wedge\delta}$, $\mathcal{B}^{\vee\delta}$, $\mathcal{C}^\delta$ in place of $g^\wedge$, $g^\vee$, $\mathcal{U}^{\wedge\triangleleft}$, $\mathcal{B}^{\vee\triangleleft}$, $\mathcal{C}^{\triangleleft}$. In this case $b \geq 1$ implies $N > 0$, so that only Type 1 systems $S_{-\infty}$, $S^{\wedge\delta}$, $S^{\vee\delta}$ need be considered, and the saddle point of $S_{-\infty}$ is $(1, N)$ and the unstable node (defined relative to the forward motion) is $(h^+_{-\infty}, 0)$.

The main conclusions so far may be summarised as follows:

Corollary 11(α). *Forward Motion.* Consider solutions $\phi(z) = (h(z), \theta(z))$ of S with $\theta(z) > 0$ of the form $\phi(z) = \phi(z; h_\Diamond, \theta_\Diamond, z_\Diamond)$ with $(h_\Diamond, \theta_\Diamond) = \pi_\Diamond \in \mathbf{U}$ and fixed $z_\Diamond \in \Re$, each defined on a 'forward' maximal interval $[z_\Diamond, z_+)$ where $z_+ = z_+(\pi_\Diamond, z_\Diamond)$.

(i) For $b > 1$, these solutions (and the corresponding initial conditions) are partitioned into three classes.

(A) Solutions with

$$h(z) > f^{\wedge}[\theta(z)] \text{ for some } z \in [z_\Diamond, z_+);$$

these solutions become unbounded as $z \uparrow z_+$. An equivalent statement is that

$$\mathcal{U}^{\triangleright}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{U}^{\wedge\triangleright} \text{ for some } z \geq z_\Diamond\}.$$

(B) Solutions with $z_+ = \infty$ and

$$f^{\wedge}[\theta(z)] > h(z) > f^{\vee}[\theta(z)], \text{ i.e. } \phi(z) \in \mathcal{C}^{\triangleright}, \text{ for all } z \in [z_\Diamond, \infty);$$

these solutions converge as $z \uparrow \infty$ to π^*_∞, i.e. to $(1, n)$ if $n > 0$, to $(h^+_\infty, 0)$ if $q > 0 > n$. An equivalent statement is that

$$\mathcal{M}^{\triangleright}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{C}^{\triangleright} \text{ for all } z \geq z_\Diamond\}.$$

(C) Solutions with $z_+ = \infty$ and

$$h(z) < f^{\vee}[\theta(z)] \text{ for some } z \in [z_\Diamond, \infty);$$

these solutions converge as $z \uparrow \infty$ to the stable node $(h^-_\infty, 0)$ of S_∞. An equivalent statement is that

$$\mathcal{B}^{\triangleright}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{B}^{\vee\triangleright} \text{ for some } z \geq z_\Diamond\}.$$

(α)(ii). For $b \leq 1$ and ρ, z^ρ chosen as in (3.51), $z_\Diamond \geq z^\rho$ and z restricted to $[z_\Diamond, z_+)$, corresponding results apply with $f^{\wedge\rho}$, $f^{\vee\rho}$, $\mathcal{U}^{\wedge\rho}$, $\mathcal{B}^{\vee\rho}$, $\mathcal{C}^\rho$ in place of $f^\wedge$, $f^\vee$, $\mathcal{U}^{\vee\triangleright}$, $\mathcal{B}^{\vee\triangleright}$, $\mathcal{C}^{\triangleright}$ cf. Prop.11(α)(ii).

(β) *Backward Motion.* Consider solutions $\phi(z) = (h(z), \theta(z))$ of S with $\theta(z) > 0$ of the form $\phi(z) = \phi(z; h_\Diamond, \theta_\Diamond, z_\Diamond)$ with $(h_\Diamond, \theta_\Diamond) = \pi_\Diamond \in \mathbf{U}$ and fixed $z_\Diamond \in \Re$, each defined on a 'backward' maximal interval $(z_-, z_\Diamond]$, where $z_- = z-(\pi_\Diamond, z_\Diamond)$.

(i) For $b \le 1$, these solutions (and the corresponding initial conditions) are partitioned into three classes.

(A) Solutions with

$$h(z) > g^{\wedge}[\theta(z)] \text{ for some } z \in (z_-, z_\Diamond];$$

these solutions become unbounded as $z \downarrow z_- > -\infty$. An equivalent statement is that

$$\mathcal{U}^{\triangleleft}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{U}^{\wedge\triangleleft} \text{ for some } z \le z_\Diamond\}.$$

(B) Solutions with $z_- = -\infty$ and

$$g^{\wedge}[\theta(z)] < h(z) < g^{\vee}[\theta(z)], \text{ i.e. } \phi(z) \in \mathcal{C}^{\triangleleft}, \text{ for all } z \in (-\infty, z_\Diamond);$$

these solutions converge as $z \downarrow -\infty$ to $\pi^*_{-\infty}$, i.e. to $(1, N)$ if $N > 0$, to $(h^-_{-\infty}, 0)$ if $q > 0 > N$. An equivalent statement is that

$$\mathcal{M}^{\triangleleft}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{C}^{\triangleleft} \text{ for all } z \le z_\Diamond\}.$$

(C) Solutions with $z_- = -\infty$ and

$$h(z) > g^{\vee}[\theta(z)] \text{ for some } z \in (-\infty, z_\Diamond];$$

these solutions converge as $z \downarrow -\infty$ to the unstable node $(h^+_{-\infty}, 0)$ of $S_{-\infty}$. An equivalent statement is that

$$\mathcal{B}^{\triangleleft}(z_\Diamond) = \{\pi_\Diamond \in \mathbf{U} \colon \phi(z; \pi_\Diamond, z_\Diamond) \in \mathcal{B}^{\vee\triangleleft} \text{ for some } z \le z_\Diamond\}.$$

(β)(ii). For $b \ge 1$, δ, z^δ chosen as in (3.43), $z \le z^\delta$ and z restricted to $(z_-, z_\Diamond)$, corresponding results apply with $g^{\wedge\delta}$, $g^{\vee\delta}$, $\mathcal{U}^{\wedge\delta}$, $\mathcal{B}^{\vee\delta}$, $\mathcal{C}^{\delta}$ in place of $g^{\wedge}$, $g^{\vee}$, $\mathcal{U}^{\vee\triangleleft}$, $\mathcal{B}^{\vee\triangleleft}$, $\mathcal{C}^{\triangleleft}$, cf. Prop.11($\beta$)(ii).

Corollary 11 has assumed a fixed $z_\Diamond \in \Re$, but it is possible to state a 'three-dimensional' version with $z_\Diamond$ variable. Note also that the preceding results yield some bounds for a star solution in addition to those stated in Corollary 8. In particular, taking into account that $f^\wedge(\theta) < h^{\wedge+}$ for all $\theta > 0$, the path of a star solution must lie in the 'box' $\{0 < h < h^{\wedge+}, 0 < \theta < \nu\}$.[3]

Remark (1). Solutions of S with $\theta > 0$ which become unbounded as $z \uparrow$ ($z \downarrow$) also converge, the possible limits being those identified in Section 3 for solutions of $\bar{S}$. Briefly, we have $\theta(z) \uparrow \infty$ on a final interval as $z \uparrow z_+$ ($z \downarrow z_-$) in all these cases. For solutions which become unbounded as $z \uparrow$, we have $h(z_+) = \infty$ with $z_+ < \infty$. For solutions which become unbounded as $z \downarrow z_-$, one of the following types of behaviour on a final interval are possible: $h(z) \downarrow 0$, $z_- = -\infty$; $h(z) \uparrow 0$, $z_- = -\infty$. The arguments are similar to those given in Section 3, but I have not worked out in detail the correspondence between parameter values and the various possible limits. These arguments do not affect the existence proofs for solutions of b.v.p.s.

Remark (2). Corollary 11 can be interpreted as stating that, in each case considered, Class B 'separates' the solutions in Class A from those in Class C. The next two Propositions show that $\mathcal{M}^{\triangleright}(z_\Diamond)$ and $\mathcal{M}^{\triangleleft}(z_\Diamond)$ are separatrices in a more usual sense, namely that they are single non-stationary paths 'behaving topologically abnormally in comparison with neighbouring paths', cf. Lefschetz [1975] p.223. Furthermore, these paths, restricted to the positive quadrant of $\Re^2$, are graphs of monotonic functions of θ, decreasing in the case of $\mathcal{M}^{\triangleright}(z_\Diamond)$, increasing in the case of $\mathcal{M}^{\triangleleft}(z_\Diamond)$. The fact that these

[3]Prop.11 and its proof can be extended to allow for solutions of S with $\theta = 0$. Choosing $\mathbf{U} = \{\theta \geq 0\}$ and considering the sets defined in eqns.(4.2) as subsets of this domain, results about the existence of limits of solutions and the assignment of sets of solutions to the available limit points are obtained as in the main text. However it is of interest to consider directly the behaviour of solutions of S with $\theta = 0$ and to relate this to corresponding results for systems $\bar{S}$, in particular S_∞ and $S_{-\infty}$, noted in Section 3, fn.11. For brevity we consider only the case corresponding to Prop.11(α)(i), i.e., $b > 1$, $z_\Diamond \leq z \uparrow z_+$, S_∞ of either Type, and take it as known that a solution of S defined on a forward unbounded interval converges to some limit, finite or infinite.

The system S, restricted to $\{\theta = 0\}$, reduces to the equation

$$(*) \qquad h' = F(h, 0, z) = F_\infty(h, 0) + (2/\sigma^2)[M/b - hA]$$

see (0.1) and (3.1). Since $F > F^\wedge > 0$ for $h > h^{\wedge+}$ with S_∞ of either Type, cf. Figs. 3 & 4(i,ii,iii), it is clear that any solution $h(z)$ of (*) which ever enters the interval $(h^{\wedge+}, \infty)$ increases thereafter without bound, so $h(z_+) = \infty$ (and $z_+ < \infty$ as in Section 3). There are no unbounded decreasing solutions of (*) because both $F_\infty(h, 0)$ and $M/b - hA$ are positive for sufficiently large negative z. For bounded solutions of (*) we have $z_+ = \infty$, and it follows from Prop.2(ii) that the candidates for limits are the points corresponding to stationary solutions of S_∞ lying on the axis $\{\theta = 0\}$, i.e., the points h_∞^- and h_∞^+. Now h_∞^- corresponds to the stable node of S_∞ (for either Type), so that by Prop.2(iii) there is an open interval containing h_∞^- such that any solution of (*) entering this interval converges to h_∞^-. As to h_∞^+, this point corresponds to the unstable node of S_∞ if S_∞ is Type 1, with $h_\infty^+ > 1$; but if S_∞ is Type 0, then h_∞^+ corresponds to the saddle point of S_∞, with $1 > h_\infty^+ > 1/b$ (or the saddle-node in case $h_\infty^+ = 1$, as in Fig.5). In any case we obtain $F_\infty(h^+, 0) = 0$ and $M/b - h^+A < 0$, implying that any solution of (*) reaching $h(z) = h_\infty^+$ at any finite z can be continued locally with $h'(z) < 0$ and so cannot converge to h_∞^+. Thus all solutions of (*) which remain bounded converge to h_∞^-.

paths are continuous and intersect will be the essence of the proof of Theorem 4.

PROPOSITION 12 (Existence Lemma for Special Starts, $b > 1$).

(α) *Forward Motion.* For fixed $z_\Diamond \in \Re$, and each $\theta = \theta_\Diamond > 0$, there is at least one $h = h_\Diamond$ such that $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ is the start of a forward special solution. The values of $\theta_\Diamond$ for which there is a positive $h_\Diamond$ with this property form an interval $(0, \theta_+)$, where $\theta_+ = \theta_+(z_\Diamond)$. For $\theta_\Diamond$ in this interval, $h_\Diamond$ is unique and the function $h_\Diamond = f(\theta_\Diamond, z_\Diamond)$, or simply $h = f(\theta)$, defined by this property is continuous, strictly decreasing in θ and satisfies

(4.5) $\quad f^\wedge(\theta) > f(\theta) > f^\vee(\theta)$. Also

(4.5a) $\quad 0 < \theta_+^\vee \leq \theta_+ \leq \theta_+^\wedge \leq \infty$ and $f(\theta_+) = 0$ in all cases;

$\theta_+ = \infty$ if $m \geq \psi_0'$; $\quad \theta_+ \leq \infty$ if $0 \leq m < \psi_0'$; $\quad \theta_+ < \infty$ if $m < 0$.

(β) *Backward Motion* For δ, z^δ chosen as in (3.43) and fixed $z_\Diamond \leq z^\delta$, there is for each $\theta = \theta_\Diamond > 0$, at least one $h = h_\Diamond$ such that $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ is the start of a backward special solution. The values of $\theta_\Diamond$ for which there is a positive $h_\Diamond$ with this property form an interval (θ_-, ∞), where $\theta_- = \theta_-(z_\Diamond)$. For $\theta_\Diamond$ in this interval, $h_\Diamond$ is unique and the function $h_\Diamond = g(\theta_\Diamond, z_\Diamond)$, or simply $h = g(\theta)$, is continuous, strictly increasing with $g(\infty) = g(\infty, z_\Diamond) = \infty$ and satisfies

(4.6) $\quad g^{\wedge\delta}(\theta) < g(\theta) < g^{\vee\delta}(\theta)$. Also

(4.6a) $\quad 0 \leq \theta_-^{\vee\delta} \leq \theta_- \leq \theta_-^{\wedge\delta}$ and $\theta_- \cdot g(\theta_-) = 0$ in all cases;

$\theta_- > 0$ if $m > \psi_0'$; $\quad \theta_- \geq 0$ if $m = \psi_0'$; $\quad \theta_- = 0$ if $m < \psi_0'$;

$g(\theta_-) = 0$ if $m \geq \psi_0'$; $\quad g(\theta_-) > 0$ if $m < \psi_0'$.

PROOF. (α) We fix $z_\Diamond$ throughout this proof and consider only the forward motion, for $z > z_\Diamond$, defined by S. Often we omit $z_\Diamond$ from the notation, also the superscript $\triangleright$ from the symbols in (4.2 a,b), so that $\mathcal{U}^\wedge = \mathcal{U}^{\wedge\triangleright}$, $\mathcal{B}^\vee = \mathcal{B}^{\vee\triangleright}$, $\mathcal{C} = \mathcal{C}^\triangleright$. Note that these are open subsets of $\{\theta > 0\}$ and of $\Re^2$. For given $z > z_\Diamond$, we denote by $\phi_z^\triangleright = \phi_z$ the transformation

$$\pi_\Diamond \mapsto \phi(z; \pi_\Diamond, z_\Diamond), \quad \text{where } \pi_\Diamond = (h_\Diamond, \theta_\Diamond),\ \phi(z) = (h(z), \theta(z)),$$

whenever this is defined; bear in mind that a solution path which enters $\mathcal{U}^\wedge$ or $\mathcal{B}^\vee$ stays in that set for $z > z_\Diamond$ as long as the solution is defined. Now let

$$\begin{aligned}(4.7)\qquad & W^u(z) = \{\pi_\Diamond \in \mathcal{C} \colon \phi_\zeta \pi_\Diamond \in \mathcal{U}^\wedge \text{ for some } \zeta \in (z_\Diamond, z]\},\\ & W^b(z) = \{\pi_\Diamond \in \mathcal{C} \colon \phi_\zeta \pi_\Diamond \in \mathcal{B}^\vee \text{ for some } \zeta \in (z_\Diamond, z]\} = \{\pi_\Diamond \in \mathcal{C} \colon \phi_z \pi_\Diamond \in \mathcal{B}^\vee\};\\ & W^c(z) = \{\pi_\Diamond \in \mathcal{C} \colon \phi_z \pi_\Diamond \in [\![\mathcal{C}]\!]\} = \{\pi_\Diamond \in [\![\mathcal{C}]\!] \colon \phi_z \pi_\Diamond \in [\![\mathcal{C}]\!]\}\end{aligned}$$

(where $W^u(z) = W^{u\rhd}(z; z_\Diamond)$ etc. and $[\![\mathcal{C}]\!] = \mathcal{C} \cup \mathcal{M}^\wedge \cup \mathcal{M}^\vee$ denotes the closure of $\mathcal{C}$ relative to $\{\theta > 0\}$). The replacement of $\mathcal{C}$ by $[\![\mathcal{C}]\!]$ in the last equality above is permissible because relative boundary points of $\mathcal{C}$ are mapped into points of $\mathcal{U}^\wedge$ or $\mathcal{B}^\vee$. Of course, $\mathcal{C}$ is the union of the three sets in (7). Clearly $W^u(z)$ and $W^b(z)$ are open (in $\{\theta > 0\}$ and in $\Re^2$) and disjoint, and in view of the one-way passage across boundaries these sets are not empty; (more details later). On the other hand, $W^c(z) = \mathcal{C} \setminus \{W^u(z) \cup W^b(z)\}$ is relatively closed and non-empty. Now, if we let $z \uparrow$ (still keeping $z_\Diamond$ fixed) it follows from the one-way passage across boundaries that the open, disjoint sets $W^u(z)$ and $W^b(z)$ increase to open, disjoint limit sets $W^u_\infty = W^u_\infty(z_\Diamond)$ and $W^b_\infty = W^b_\infty(z_\Diamond)$. The sets $W^c(z)$ decrease to a relatively closed limit set $W^c_\infty = W^c_\infty(z_\Diamond)$, (and this set also is not empty since it has the form $\mathcal{C} \setminus \{W^u_\infty \cup W^b_\infty\}$ with $\mathcal{C}$, W^u_∞ and W^b_∞ all open and non-empty and W^u_∞, W^b_∞ disjoint).

Consider now the sections of these various sets at a fixed $\theta_\Diamond > 0$. We have $\mathcal{C}(\theta_\Diamond) = (f^\vee(\theta_\Diamond), f^\wedge(\theta_\Diamond))$, an open interval of positive length. For given z, the section $W^u(z, \theta_\Diamond)$ is open in $\Re$ (as the section of an open set), and it is contained in the open interval $\mathcal{C}(\theta_\Diamond)$. It further follows from $F > F^\wedge$ and the continuity of the various functions that a solution starting at $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ with $h_\Diamond \in \mathcal{C}(\theta_\Diamond)$ will pass into $\mathcal{U}^\wedge$ before z if $h_\Diamond$ is close enough to $f^\wedge(\theta_\Diamond)$, so $W^u(z; \theta_\Diamond)$ contains an interval of the form $(h^u(z), f^\wedge(\theta_\Diamond))$, where $h^u(z) = h^u(z; \theta_\Diamond, z_\Diamond)$. Similarly $W^b(z; \theta_\Diamond)$ is open in $\Re$ and contains an interval $(f^\vee(\theta_\Diamond), h^b(z))$, and $W^u(z; \theta_\Diamond)$ and $W^b(z; \theta_\Diamond)$ are disjoint. Since

$$(4.8)\qquad W^c(z; \theta_\Diamond) = \mathcal{C}(\theta_\Diamond) \setminus \{W^u(z; \theta_\Diamond) \cup W^b(z; \theta_\Diamond)\}$$

this set must be closed in $\Re$, bounded and non-empty. Letting $z \uparrow \infty$, it follows from the monotonicity of the various convergences that, for each of the sets $W^u(z)$, $W^b(z)$ and $W^c(z)$, the limit of the section at $\theta_\Diamond$ is the section at $\theta_\Diamond$ of the limit. In particular, $W^u(z; \theta_\Diamond)$ and $W^b(z; \theta_\Diamond)$ increase to sets $W^u_\infty(\theta_\Diamond)$ and $W^b_\infty(\theta_\Diamond)$ which are disjoint, bounded and open in $\Re$ and in $\mathcal{C}(\theta_\Diamond)$ with upper and lower endpoints $f^\wedge(\theta_\Diamond)$ and $f^\vee(\theta_\Diamond)$ of $\mathcal{C}(\theta_\Diamond)$ respectively, so $W^c_\infty(\theta_\Diamond)$ is a non-empty, closed and bounded set in $\Re$ and in $\mathcal{C}(\theta_\Diamond)$. *Now, a point $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ with $h_\Diamond \in W^c_\infty(\theta_\Diamond; z_\Diamond)$ is the start of a solution whose* *path for $z \geq z_\diamondsuit$ lies entirely in $\mathcal{C}$, so by Prop. 11(α)(i) it is the start of a forward special solution. It follows that for each $(\theta_\diamondsuit; z_\diamondsuit)$ there is at least one such start.*

If, for fixed $\theta_\diamondsuit > 0$, there is one start of a forward special solution with $h_\diamondsuit \geq 0$, then according to Cor. 10.1 this $h_\diamondsuit$ is *unique.* Since $f^\vee(\theta) < h < f^\wedge(\theta)$ for $h \in \mathcal{C}(\theta)$, this will be the case at least for $\theta = \theta_\diamondsuit$ in the interval $(0, \theta_+^\vee)$ where $f^\vee$ is positive, and here we may write $h_\diamondsuit$ as a function $h = f(\theta, z_\diamondsuit) = f(\theta)$. Consider f first as a function on a closed interval $[\theta_1, \theta_2]$ with $0 < \theta_1 < \theta_2 < \theta_+^\vee$, and note that on this interval the function takes values in the interval $[0, h^{\wedge+}]$, (and so may be considered as defining a relation in a product of compact metric spaces). The graph of the function is $W_\infty^c \cap \{\theta_1 \leq \theta \leq \theta_2\}$, which is a closed and bounded plane set, implying that the function is *continuous* on the interval $[\theta_1, \theta_2]$, see Akin [1993] p.8 or Akin [1996] p.7, and letting $\theta_1 \downarrow 0$, $\theta_2 \uparrow \theta_+^\vee$, is seen to be continuous on $(0, \theta_+^\vee)$. Obviously the function is *positive* on this interval, and the fact that it is *decreasing* is a consequence of Prop. 10(α)(i).

If $m \geq \psi_0'$, then $\theta_+^\vee = \theta_+^\wedge = \infty$ by (3.57a), so f is defined on $(0, \theta_+)$ with $\theta_+ = \theta_+^\vee = \infty$, and $f(\infty) = 0$ by (3.57) since $f^\wedge > f > f^\vee$; *in this case, the proof is complete.* Suppose that $m < \psi_0'$ and $\theta_+^\vee < \infty$. It remains true that $W_\infty^c(z_\diamondsuit)$ is the graph of a continuous simple curve, say of the form $\mathfrak{f}(h, \theta, z_\diamondsuit) = 0$ with at least one solution $h = f(\theta, z_\diamondsuit)$ for each $\theta > 0$. This follows (for example) from the facts that every forward special solution eventually enters the strip $\{0 < \theta < \theta_+^\vee\}$, and that the map ϕ_z defines for each $z > z_\diamondsuit$ a homeomorphism from $W_\infty^c(z_\diamondsuit)$ to $W_\infty^c(z)$. Further, Prop. 10(α)(i) with Cor. 10.1 ensures that values of θ for which $W_\infty^c(z_\diamondsuit)$ contains only negative values of h are not succeeded by greater values of θ with positive h; thus the values of θ for which $W_\infty^c(z_\diamondsuit)$ contains precisely one $h > 0$ form an interval $0 < \theta < \theta_+ = \theta_+(z_\diamondsuit)$ and the continuous, positive decreasing function f can be extended to this interval. Since $f^\wedge > f > f^\vee$ on the interval, it follows that $\theta_+^\vee \leq \theta_+ \leq \theta_+^\wedge$. Moreover $f(\theta_+) = 0$; indeed, we have *either* $\theta_+ = \infty$ and then the assertion follows as above, *or* $\theta_+ < \infty$ and then it follows from continuity and monotonicity of f. Finally, it follows from (3.57a) that $\theta_+ < \infty$ if $m < 0$, but it seems that in general both possibilities are open if $0 \leq m < \psi_0'$.$\|$

With the notation (4.1) et.seq, the preceding proof shows that

$$W_\infty^c(z_\diamondsuit) = \{\pi_\diamondsuit \colon \phi(z; \pi_\diamondsuit, z_\diamondsuit)\} \in \mathcal{C} \text{ for all } z \geq z_\diamondsuit = \mathcal{M}^\triangleright(z_\diamondsuit),$$

and $f(\cdot; z_\diamondsuit)$ is the graph of $\mathcal{M}^\triangleright(z_\diamondsuit)$ restricted to $\theta \in (0, \theta_+(z_\diamondsuit)$. Note also that the argument is stated in a way which avoids the need to distinguish between Type 1 and Type 0 systems, but if S_∞ is Type 0 and we choose $\{\theta \geq 0\}$ as domain for S, the function

$f(\theta)$ may be extended to $[0, \theta_+^\vee)$.

(β) The main part of the proof is similar. Briefly, we consider only solutions defined for $z \leq z_\Diamond$ where $z_\Diamond \geq z^\delta$. The forward motion $\phi^\triangleright$ is replaced by the backward motion $\phi^\triangleleft$, $\mathcal{C} = \mathcal{C}^\triangleright, \mathcal{U}^\wedge$, $\mathcal{B}^\vee$ are replaced by $\mathcal{C}^\delta, \mathcal{U}^{\wedge\delta}$ and $\mathcal{B}^{\vee\delta}$, W^c_∞ is replaced by $W^c_{-\infty}$ etc, and the roles of $f^\vee$, $\theta_+^\vee$, $f^\wedge$, $\theta_+^\wedge$, are taken over by $g^{\wedge\delta}$, $\theta_-^{\wedge\delta}$, $g^{\vee\delta}$, $\theta_-^{\vee\delta}$. The proof then follows much the same lines up to the point where it is established that $W^c_{-\infty}(z_\Diamond) = \mathcal{M}^\triangleleft(z_\Diamond)$ can be represented, at least on $(\theta_-^{\wedge\delta}, \infty)$, by a continuous, positive, increasing function g with $g^{\wedge\delta} < g < g^{\vee\delta}$, hence $g(\infty) = \infty$. (Minor changes are needed to allow for differences between properties of $f^\vee$ and $g^{\wedge\delta}$, $\theta_+^\vee$ and $\theta_-^{\wedge\delta}$ etc. resulting from Props. 6 and 7).

In the last paragraph of the proof, the roles of the inequalities involving $m - \psi_0'$ are interchanged. More precisely, if $m < \psi_0'$, then $\theta_-^{\vee\delta} = \theta_-^{\wedge\delta} = 0$ and $g^{\vee\delta}(0) > g^{\wedge\delta}(0) > 0$ by (3.58–59), so that g is defined on $(0, \infty)$ with a limit $g(0) > 0$; in this case, the proof is complete. If $m > \psi_0'$, then $0 < \theta_-^{\vee\delta} < \theta_-^{\wedge\delta}$ and $g^{\vee\delta}(\theta_-^{\vee\delta}) = g^{\wedge\delta}(\theta_-^{\wedge\delta}) = 0$ by (3.58–59). The graph of $W^c_{-\infty}$ is again a continuous simple curve, say $\mathfrak{g}(h, \theta, z_\Diamond) = 0$ with at least one solution $h = g(\theta, z_\Diamond)$ for each $\theta > 0$, and the representation $h = g(\theta)$ with $g > 0$ can be extended to a maximal interval (θ_-, ∞) where $\theta_- = \theta_-(z_\Diamond)$, satisfying $0 < \theta_-^{\vee\delta} < \theta_- < \theta_-^{\wedge\delta}$ and $g(\theta_-) = 0$. The case $m = \psi_0'$ needs separate consideration and one apparently has only $0 = \theta_-^{\vee\delta} \leq \theta_- < \theta_-^{\wedge\delta}$ and $g(\theta_-) = 0$.$\|$ Now $W^c_{-\infty}(z_\Diamond) = \mathcal{M}^\triangleleft(z_\Diamond)$, and $g(\cdot, z_\Diamond)$ is the graph of $\mathcal{M}^\triangleleft(z_\Diamond)$ restricted to $\theta \in (\theta_-(z_\Diamond), \infty)$.

Proposition 13 (Existence Lemma for Special Starts, $b \leq 1$).

(α) *Forward Motion.* For ρ, z^ρ chosen as in (3.51) and fixed $z_\Diamond \geq z^\rho$, there is for each $\theta = \theta_\Diamond > 0$, at least one $h = h_\Diamond$ such that $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ is the start of a forward special solution. The values of $\theta_\Diamond$ for which there is a positive $h_\Diamond$ with this property form an interval $(0, \theta_+)$, where $\theta_+ = \theta_+(z_\Diamond)$. For $\theta_\Diamond$ in this interval, $h_\Diamond$ is unique and the function $h_\Diamond = f(\theta_\Diamond, z_\Diamond)$, or simply $h = f(\theta)$, is continuous, strictly decreasing and satisfies

(4.9) $\quad f^{\wedge\rho}(\theta) > f(\theta) > f^{\vee\varrho}(\theta)$. Also

(4.9a) $\quad 0 < \theta_+^{\vee\rho} \leq \theta_+ \leq \theta_+^{\wedge\varrho} \leq \infty$ and $f(\theta_+) = 0$ in all cases;

$\theta_+ = \infty$ if $m > 0$; $\quad \theta_+ \leq \infty$ if $m = 0$; $\quad \theta_+ < \infty$ if $m < 0$.

(β) *Backward Motion.* For each fixed $z_\Diamond \in \Re$, and each $\theta = \theta_\Diamond > 0$, there is at least one $h = h_\Diamond$ such that $(h_\Diamond, \theta_\Diamond, z_\Diamond)$ is the start of a backward special solution. The values of $\theta_\Diamond$ for which there is a positive $h_\Diamond$ with this property form an interval (θ_-, ∞), where $\theta_- = \theta_-(z_\Diamond)$. For $\theta_\Diamond$ in this interval, $h_\Diamond$ is unique and the function $h_\Diamond = g(\theta_\Diamond, z_\Diamond)$, or simply $h = g(\theta)$, defined by this property is continuous, strictly increasing and satisfies

(4.10) $\qquad g^\wedge(\theta) < g(\theta) < g^\vee(\theta).\quad$ Also

(4.10a) $\qquad 0 \le \theta_-^\vee \le \theta_- \le \theta_-^\wedge$ and $\theta_- \cdot g(\theta_-) = 0,$ in all cases;

$\theta_- > 0$ if $m > \psi_0'$; $\quad \theta_- \ge 0$ if $0 < m \le \psi_0'$; $\quad \theta_- = 0$ if $m \le 0$

$g(\theta_-) = 0$ if $m \ge \psi_0'$; $\quad g(\theta_-) \ge 0$ if $0 \le m < \psi_0'$; $\quad g(\theta_-) > 0$ if $m < 0$.

Proof. (α) and (β). This is analogous to the proof of Prop. 12 and will not be set out in detail. The best symmetry is obtained if the backward and forward proofs for $b \le 1$ follow the forward and backward proofs for $b > 1$ respectively. Thus the main part of the proof of Prop. 13(β) is like that of 12(α), replacing $\phi^\triangleright$ by $\phi^\triangleleft$, $\mathcal{C}^\triangleright$ by $\mathcal{C}^\triangleleft$ etc, the roles of $f^\vee$, $\theta_+^\vee$, $f^\wedge$, $\theta_+^\wedge$ being taken over by $g^\wedge$, $\theta_-^\wedge$, $g^\vee$, $\theta_-^\vee$ (with minor changes taking into account Props. 6 and 7). In the last paragraph of the proof, the distinction between cases with $m \ge \psi_0'$, $m < \psi_0'$ is replaced by a distinction between $m \le 0$, $m > 0$. (If $m \le 0$, then $\theta_-^\wedge = \theta_-^\vee = 0$ and the proof is completed with $\theta_- = 0$ immediately, taking into account Prop. 7; but if $m > 0$ an extension argument is needed.) If $S_{-\infty}$ is Type 0, and we choose $\{\theta \ge 0\}$ as domain for S, $\theta_\Diamond > 0$ may be replaced by $\theta_\Diamond \ge 0$ in the first sentence of 13(β), and (θ_-, ∞) by $[\theta_-, \infty)$ in the second sentence. Also, for fixed $z_\Diamond$, $W_{-\infty}^c(z_\Diamond) = \mathcal{M}^\triangleleft(z_\Diamond)$ and $g(\cdot, z_\Diamond)$ is the graph of $\mathcal{M}^\triangleleft(z_\Diamond)$ restricted to $(\theta_-(z_\Diamond), \infty)$.

Again, the main part of the proof of 13(α) is like that of 12(β), with $\phi^\triangleright$ considered only for $z \ge z_\Diamond$, where $z_\Diamond \ge z^\rho$, $\mathcal{C}^\delta$ etc. replaced by $\mathcal{C}^\rho$ etc, and the roles of $g^{\wedge\delta}$, $\theta_-^{\wedge\delta}$, $g^{\vee\delta}$, $\theta_-^{\vee\delta}$ taken over by $f^{\vee\rho}$, $\theta_+^{\vee\rho}$, $f^{\wedge\rho}$, $\theta_+^{\wedge\rho}$. In the last paragraph, the distinctions among cases with $m < \psi_0'$, $m > \psi_0'$ and $m = \psi_0'$ are replaced by $m > 0$, $m < 0$, $m = 0$. (If $m > 0$, then $\theta_+^{\vee\rho} = \theta_+^{\wedge\rho} = \infty$, and one gets $\theta_+ = \infty$ immediately, otherwise an extension argument is needed; the 'borderline' case $m = 0$ needs special consideration).‖ Now $W_\infty^c(z_\Diamond) = \mathcal{M}^\triangleright(z_\Diamond)$ and $f(\cdot, z_\Diamond)$ is the graph of $\mathcal{M}^\triangleright(z_\Diamond)$ restricted to $(\theta, \theta_+(z_\Diamond))$.

Proof of theorem 4.

This can now be completed fairly trivially. Let $b > 1$, choose δ and $z_\Diamond \le z^\delta$ as in (3.43) and define f, θ_+, g, θ_- as in Prop. 12. Suppose first that $m \ge \psi_0'$. In this case, Prop. 12(α), eqs. (5–5a), yield $\theta_+ = \infty$, $f(\infty) = 0$, with $f \downarrow$ on $(0, \infty)$; also $f(\nu) < f^\wedge(\nu)$. On the other hand, eqs. (6–6a) yield $\theta_- \ge 0$, $g(\theta_-) = 0$ with $g \uparrow$ on (θ_-, ∞), also $g(\nu) > g^{\wedge\delta}(\nu) > g^\wedge(\nu) = 1 = f^\wedge(\nu)$ using (3.46c), hence $\theta_- < \nu$. It follows that

$$f(\theta_-) > g(\theta_-) = 0, \quad f(\theta) < g(\theta) \text{ for } \nu \le \theta < \infty,$$

and since $f \downarrow$ and $g \uparrow$ there is precisely one intersection of the graphs of f and g in the interval (θ_-, ∞). This intersection defines a point $(h^*(z_\Diamond), \theta^*(z_\Diamond)) = (h^*_\Diamond, \theta^*_\Diamond)$ satisfying

$$h^*_\Diamond = f(\theta^*_\Diamond) = g(\theta^*_\Diamond) > 0, \tag{4.11}$$

and clearly $\theta_- < \theta^*_\Diamond < \nu$. This point is the start of both a forward and a backward special solution and so is a start of a star solution. It is the only point with these properties, since for $\theta \leq \theta_-$ any point (h, θ) which is the start of a b.s.s. has $h \leq 0$ while any point which is the start of a f.s.s. has $h > 0$. This completes the proof for $m \geq \psi'_0$.

Now let $m < \psi'_0$. Here (5–5a) yield $\theta_+ \leq \infty$, $f(\theta_+) = 0$ with $f \downarrow$ on $(0, \theta_+)$. On the other hand, (6–6a) yield $\theta_- = 0$, $g(\theta_-) > 0$ with $g \uparrow$ on $(0, \infty)$, also $g(\nu) > g^\wedge(\nu) = 1$. Now (3.48–9) together with $g(0) \leq g^{\vee\delta}(0)$ and $f^\vee(0) \leq f(0)$ yields $g(0) < f(0)$ in all cases. Also $g(\theta_+) > 0 = f(\theta_+)$, so that there is precisely one intersection of the curves f and g in the interval $(0, \theta_+)$, defining a point $(h^*_\Diamond, \theta^*_\Diamond)$ which satisfies (11). (Moreover $\theta^* < \nu$ as before; if $\theta_+ \leq \nu$ this is obvious, and if $\nu \leq \theta_+$ it follows from $f(\nu) < f^\wedge(\nu) = 1 < g^{\wedge\delta}(\nu) < g(\nu)$ as before.) Once again this point is the start of both a f.s.s. and a b.s.s.; and it is the only such point, since for $\theta > \theta_+$ a point which is the start of a f.s.s. has $h \leq 0$, while a point which is the start of a b.s.s. has $h > 0$.

If $b \leq 1$, we choose ρ and $z_\Diamond = z^\rho$ as in (3.51) ff. and try to imitate the preceding argument with f, θ_+, g, θ_- as in Prop. 13 and other replacements as in (3.60–61); for brevity, we shall merely note some additional minor changes. We now have $f(\theta_+) = 0$ in all cases by Prop. 13. If $m \geq \psi'_0$, Prop. 13 further yields $\theta_+ = \infty$, $\theta_- \geq 0$, $g(\theta_-) = 0$, and the proof is like that for $b > 1$, $m \geq \psi'_0$. If $m \leq 0$, Prop. 13 yields $\theta_+ \leq \infty$, $\theta_- = 0$, $g(\theta_-) \geq 0$, and the proof is like that for $b > 1$, $m < \psi'_0$. In particular, the inequality $g(0) < f(0)$ now follows from (3.48–49) with the substitutions (3.60–61) together with $g(0) < g^\vee(0)$, $f^{\vee\rho}(0) < f(0)$. Finally, if $0 < m < \psi'_0$, Prop. 13 yields $\theta_+ = \infty$, $\theta_- \geq 0$, $g(\theta_-) \geq 0$ and $\theta_- \cdot g(\theta_-) = 0$. It is then necessary to distinguish between cases with $\theta_- > 0$, $g(\theta_-) = 0$ and those with $\theta_- = 0$, $g(\theta_-) \geq 0$; in the former case, the proof is as for $m \geq \psi'_0$, in the latter as for $m \leq 0$.‖

REMARKS: (3) Props. 12 and 13 and their proofs can be simplified if both $S_{-\infty}$ and $S_{+\infty}$ are Type 1. In Prop. 12, it is assumed that $b > 1$, so that $S_{-\infty}$ is always Type 1; if S_∞ is also Type 1, then in the statement of 12(β) we may replace z^δ by an arbitrary $z_\Diamond \in \Re$ and also replace $g^{\wedge\delta}$, $g^{\vee\delta}$ by $g^\wedge$, $g^\vee$. Similarly, in Prop. 13, it is assumed that $b \leq 1$, so that S_∞ is always Type 1; and if $S_{-\infty}$ is also Type 1, then in the statement of 13(α) we may replace z^ρ by an arbitrary $z_\Diamond \in \Re$ and also replace $f^{\wedge\rho}$, $f^{\vee\rho}$ by $f^\wedge$, $f^\vee$. Thus

Props. 12 and 13 are equivalent if we consider only Type 1 Systems and allow any $b > 0$.

Also, it is shown in the proof of Theorem 4 that, for a *suitably chosen* fixed $z_\Diamond$ (far left if $b > 1$, far right if $b < 1$), the curves $h = f(\theta, z_\Diamond)$ and $h = g(\theta, z_\Diamond)$ meet in a single point $(h^*(z_\Diamond), \theta^*(z_\Diamond))$ satisfying $h^*(z_\Diamond) > 0$ and $\theta_-(z_\Diamond) < \theta^*(z_\Diamond) < \theta_+(z_\Diamond)$. If both $S_{-\infty}$ and S_∞ are Type 1, then $z_\Diamond$ may be chosen *arbitrarily* in $\Re$ and $g^{\wedge\delta}$, $g^{\vee\delta}$, $f^{\wedge\rho}$, $f^{\vee\rho}$ may be replaced in the proof by $g^\wedge$, $g^\vee$, $f^\wedge$, $f^\vee$, with any $b > 0$. In this case, the *uniqueness* of the star solution follows immediately from the fact that $f(\theta; z_\Diamond)$ is decreasing and $g(\theta; z_\Diamond)$ is increasing on the θ-interval where both functions are positive. Of course, whatever the Types, once it is shown that for a *particular* $z_\Diamond$ there is a unique point $(h^*(z_\Diamond), \theta^*(z_\Diamond), z_\Diamond)$ with $h^*(z_\Diamond) > 0$ and $\theta_-(z_\Diamond) < \theta^*(z_\Diamond) < \theta_+(z_\Diamond)$ which is the start of a star solution ϕ^*, then (taking into account Prop.8) the same is true for *arbitrary* $z_\Diamond \in \Re$.

(4) The method of proof for Props. 12–13 is loosely related to the topological method of investigating the asymptotic behaviour proposed by Wazewski [1947].

REFERENCES

Abraham, R. and Robbin, J. [1967], *Transversal Mappings and Flows.* Amsterdam, W. A. Benjamin Inc.

Akin, E. [1993], *The general topology of dynamical systems.* American Mathematical Society.

Akin, E. [1996], "Dynamical Systems: The Topological Foundations", in B. Aulbach & F. Colonius, *Six Lectures on Dynamical Systems.* World Scientific.

Chillingworth, D. R. J. [1976], *Differential topology with a view to applications.* London, Pitman.

Coddington, E. A. and Levinson, N. [1955], *Theory of Ordinary Differential Equations.* New York, McGraw-Hill.

Foldes, L.P. [2001], "The Optimal Consumption Function in a Brownian Model of Accumulation — Part A: The Consumption Function as Solution of a Boundary Value Problem". *Journal of Economic Dynamics and Control* 25, pp.1951–1971, also (with corrections) `http://personal.lse.ac.uk/foldes`. An earlier version appeared as *Discussion Paper No.* TE/96/297*, STICERD, London School of Economics.*

Foldes, L.P. [1996], "The Optimal Consumption Function in a Brownian Model of Accumulation — Part B: Existence of Solutions of Boundary Value Problems", *Discussion Paper No.* TE/96/310*, STICERD, London School of Economics.* (An earlier version of the present work.)

Hirsch, M.W. [1976], *Differential Topology.* Springer-Verlag.

Hirsch, M.W. [1984], "The Dynamical Systems Approach to Differential Equations", *Bulletin of the American Mathematical Society (N.S.)* **11**, 1–64.

Kamke, E. [1932], "Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II", *Acta Mathematica* **58**, 57–85.

Kamke, E. [1943], *Differentialgleichungen — Lösungsmethoden und Lösungen.* Leipzig, Becker & Erler.

Lefschetz, S. [1977], *Differential Equations: Geometric Theory,* 2nd edition, Dover NY.

Markus, L. [1956], "Asymptotically autonomous differential systems", in S. Lefschetz ed., *Contributions to the Theory of Non-linear Oscillations Vol.* III, Princeton U.P.

Natanson, I.P. [1955] *Theory of Functions of a Real Variable, Vol.* I, Ungar, New York.

Nemytskii, V.V. and Stepanow, V.V. [1960], *Qualitative Theory of Differential Equations.* Princeton U.P.

Opial, Z. [1960], "Sur la dépendence d'un système d'équations différentielles de leur seconds membres. Application aux systèmes presque autonomes." *Annales Polonici Mathematici* **VIII**, 75–89.

Palis, J. and de Melo, W. [1982], *Geometric Theory of Dynamical Systems — An Introduction,* New York, Springer-Verlag.

Smith, H.L. [1988], "Systems of Ordinary Differential Equations Which Generate an Order Preserving Flow. A Survey of Results." *SIAM Review* **30**, 87–114.

Smith, H.L. [1995], *Monotone Dynamical Systems.* Mathematical Surveys and Monographs Vol. 41, American Mathematical Society.

Ważewski, T. [1947], "Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations differentielles ordinaires." *Annales de la Société Polonaise de Mathématique* **20**. 279–313.

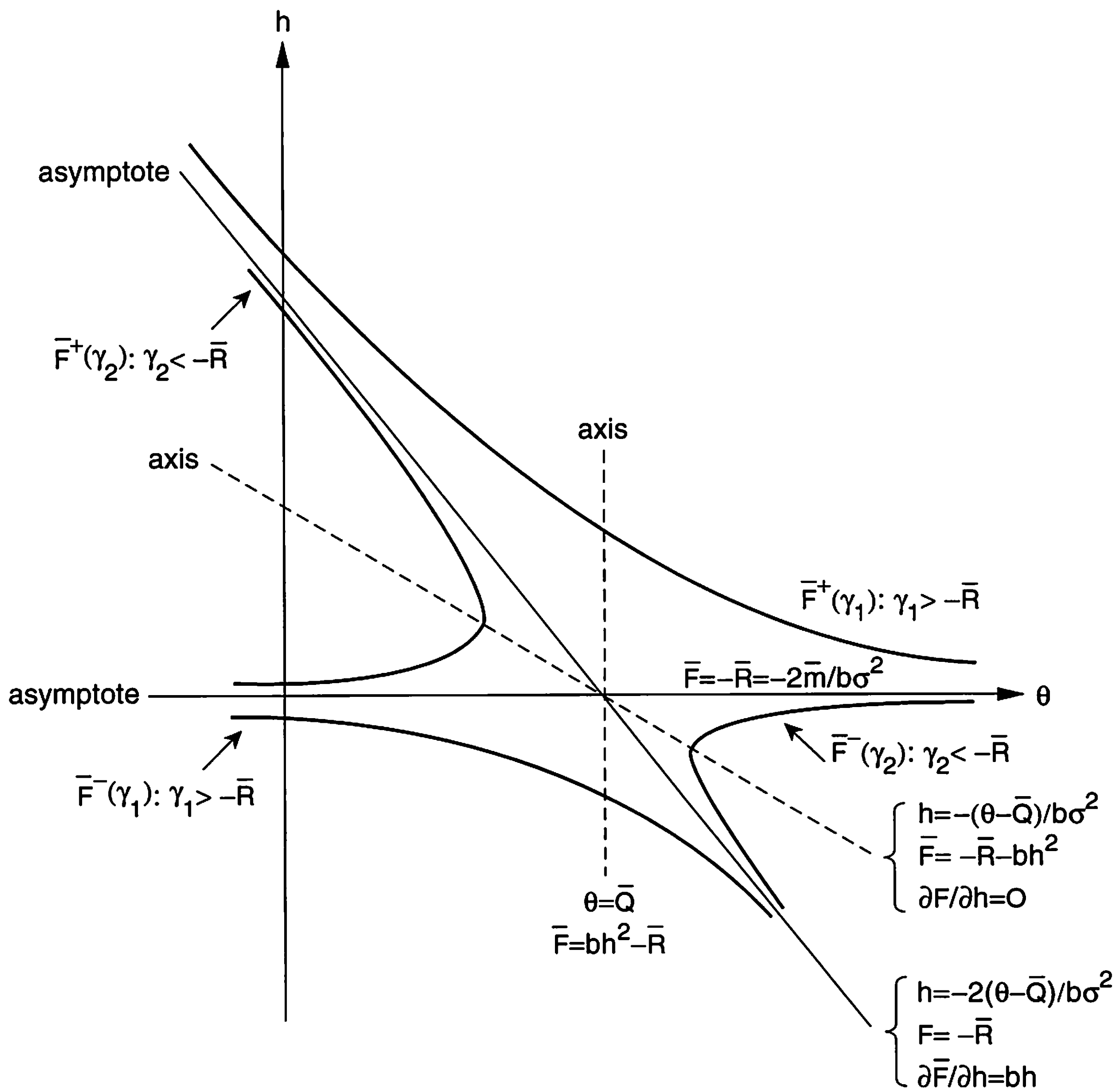

Figure 1: Contours of $\bar{F}(h,\theta)$

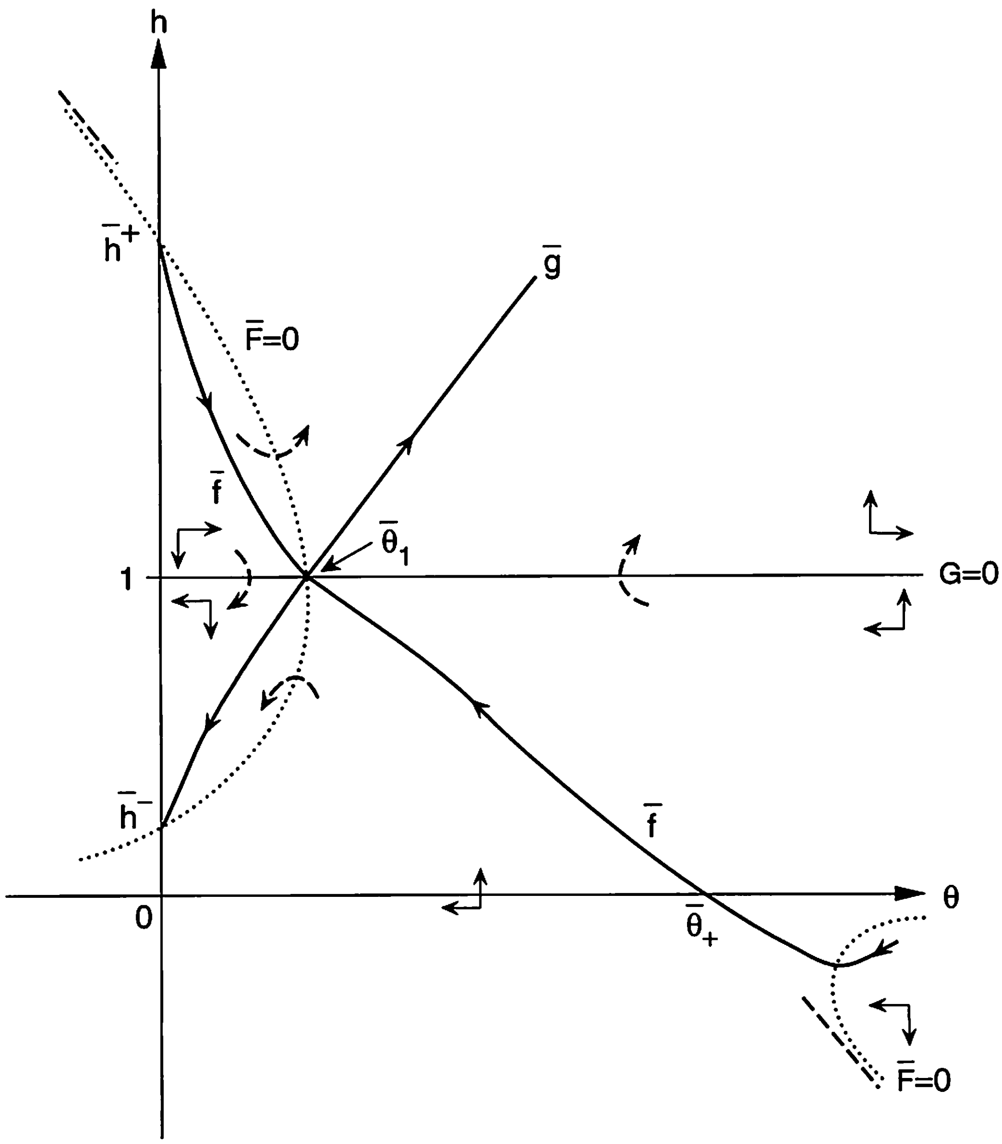

Figure 2(i): 3–Parameter System, Type 1

$\bar{\theta}_1 > 0,\ \bar{m} < 0,\ \bar{h}^+ > 1 > \bar{h}^- > 0$

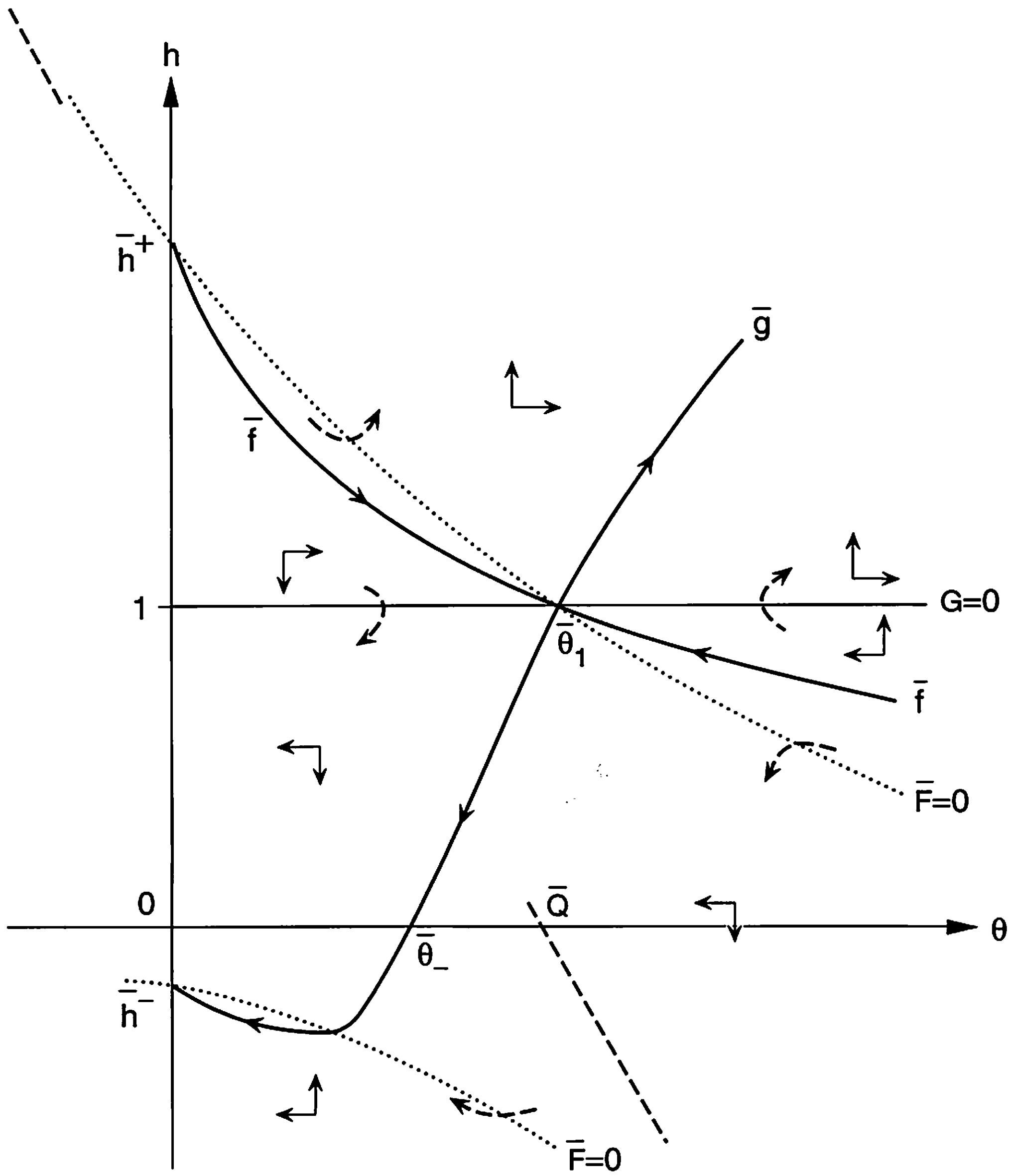

Figure 2(ii): 3–Parameter System, Type 1

$\bar{\theta}_1 > 0,\ \bar{m} > 0,\ \bar{h}^+ > 1 > 0 > \bar{h}^-$

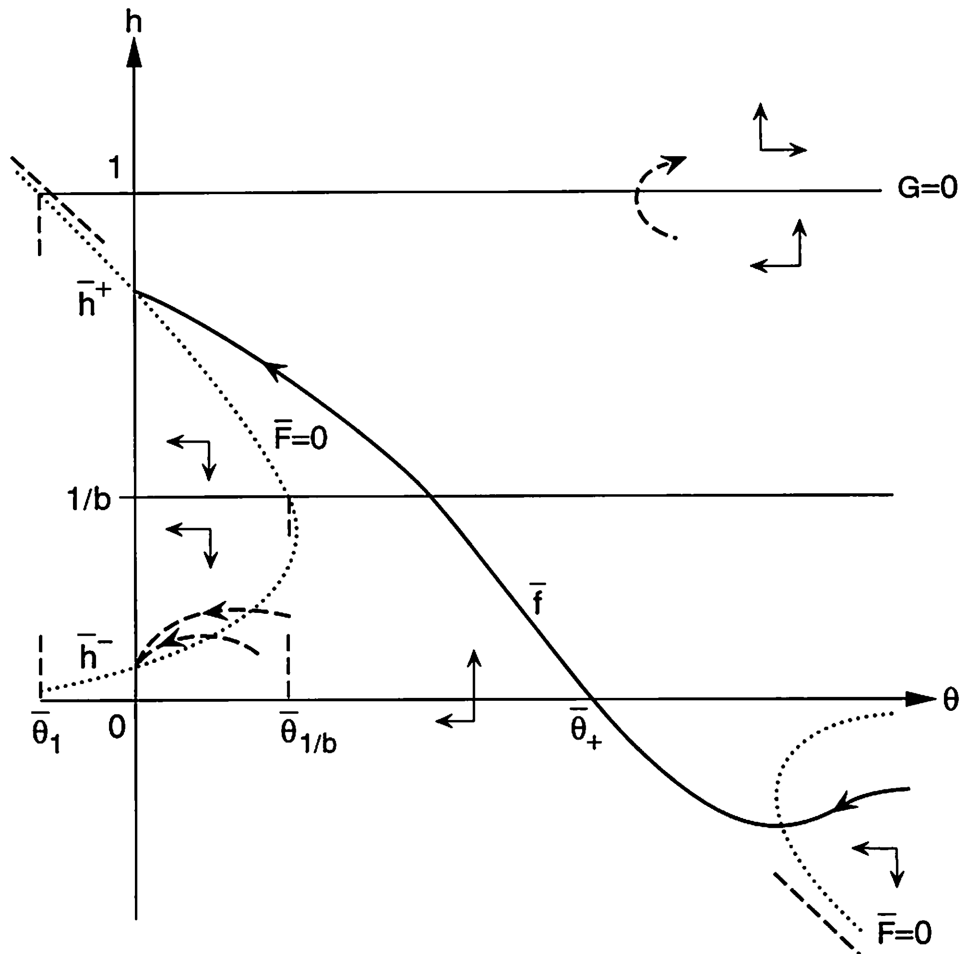

Figure 2(iii): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1,\ b > 1, \bar{m} < 0, 1 > \bar{h}^+ > 1/b > \bar{h}^- > 0$

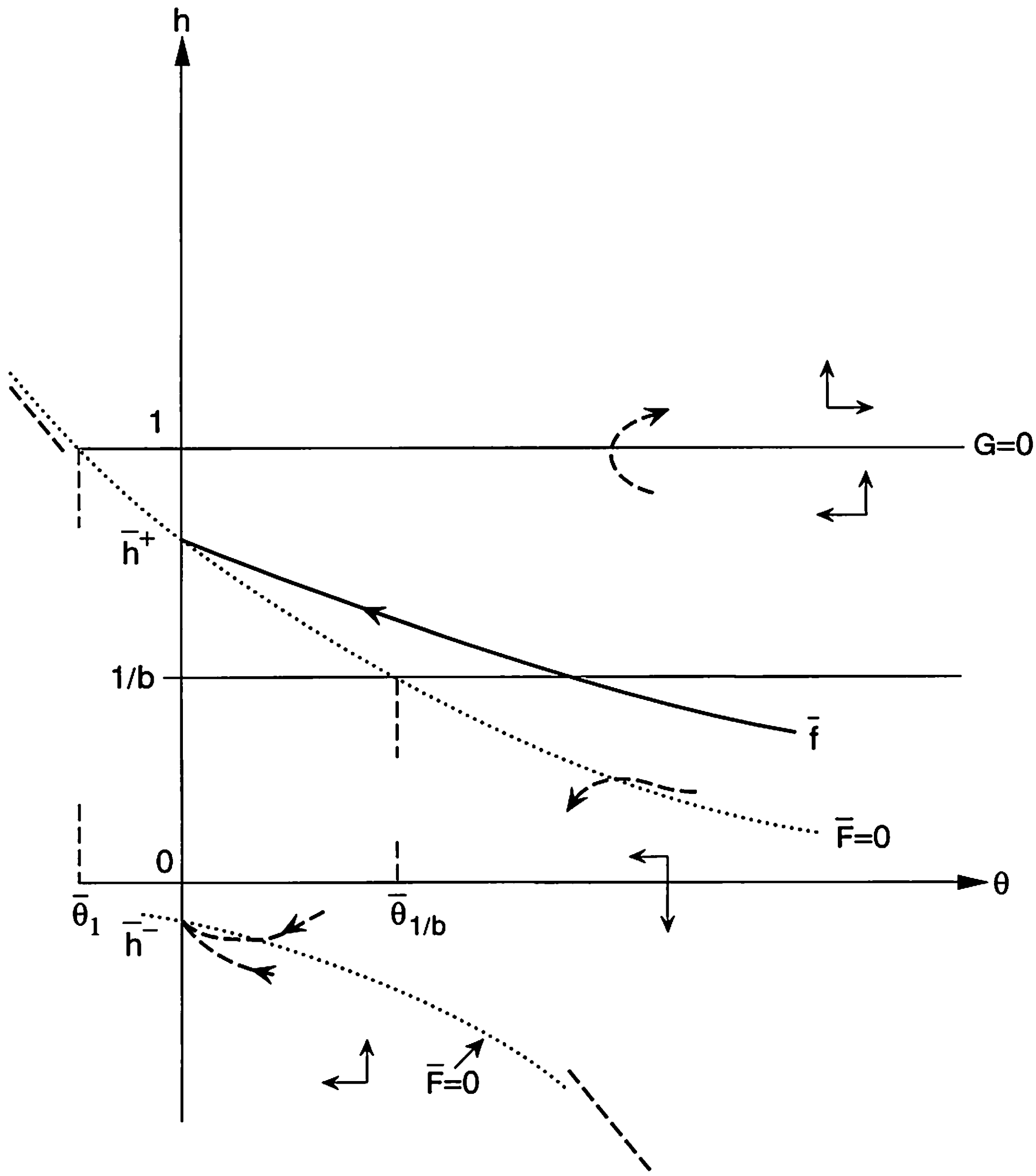

Figure 2(iv): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1, \; b > 1, \bar{m} > 0, 1 > \bar{h}^+ > 1/b > 0 > \bar{h}^-$

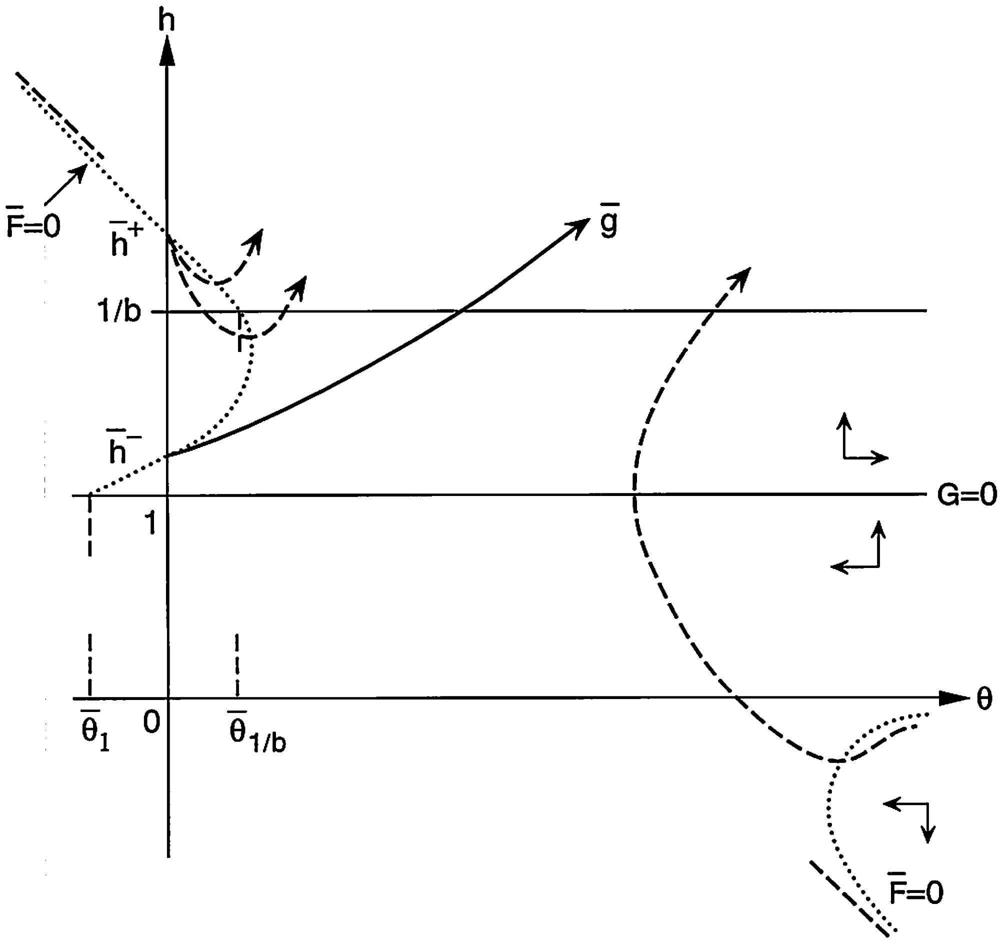

Figure 2(v): 3–Parameter System, Type 0

$\bar{\theta}_{1/b} > 0 > \bar{\theta}_1,\ b < 1, \bar{m} < 0, \bar{h}^+ > 1/b > \bar{h}^- > 1$

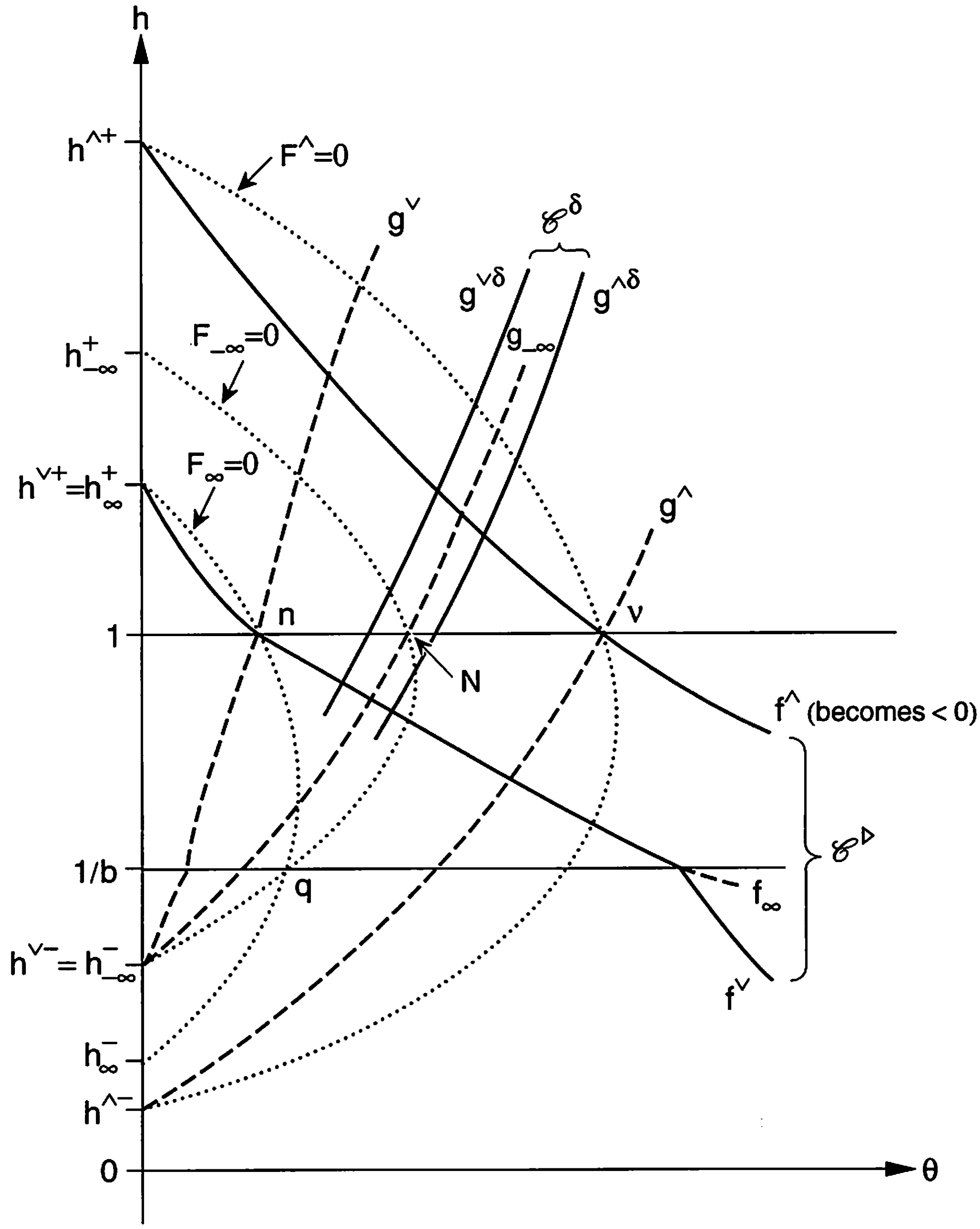

Figure 3(i): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ m < 0 < \psi_o'$

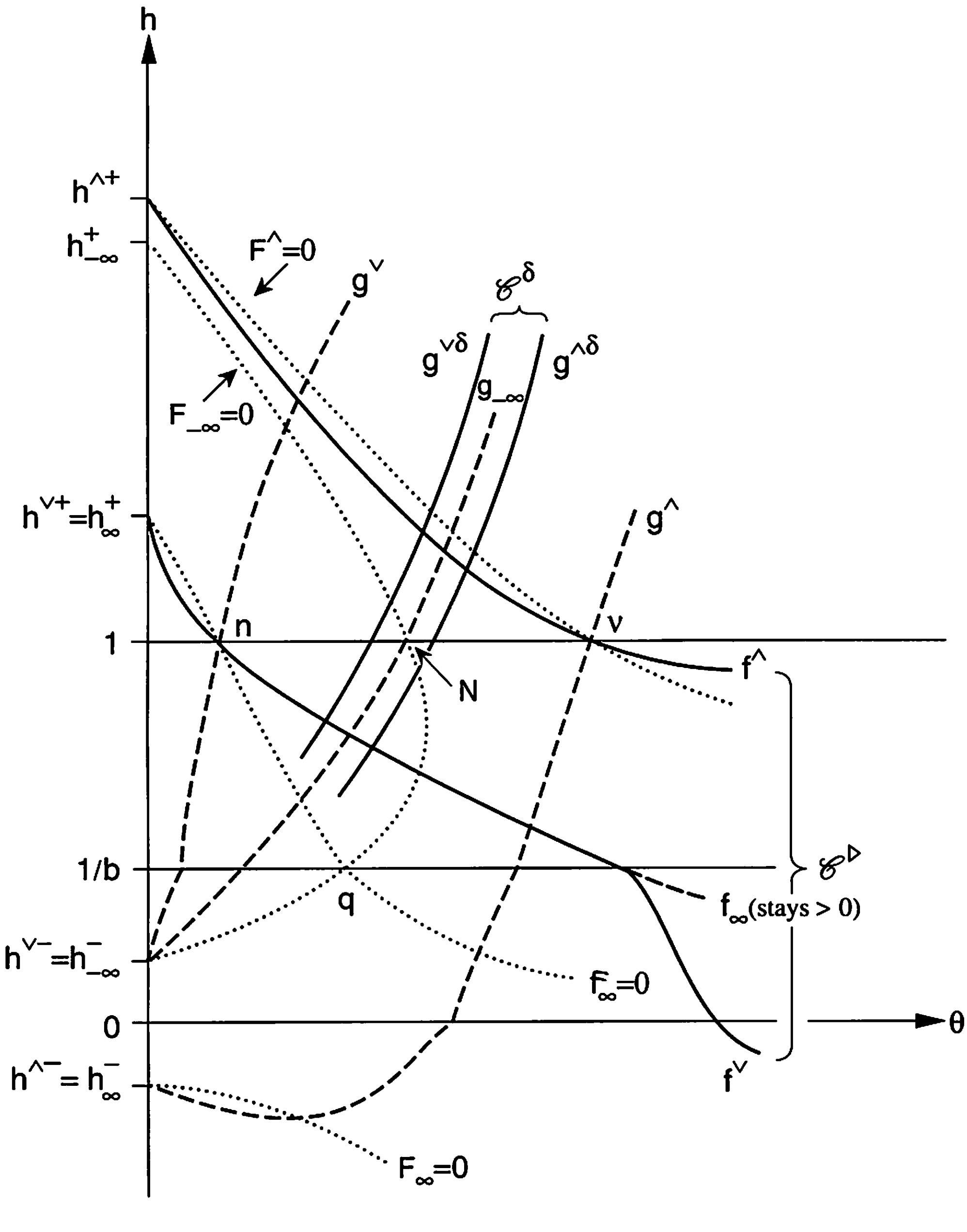

Figure 3(ii): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ 0 < m < \psi_o'$

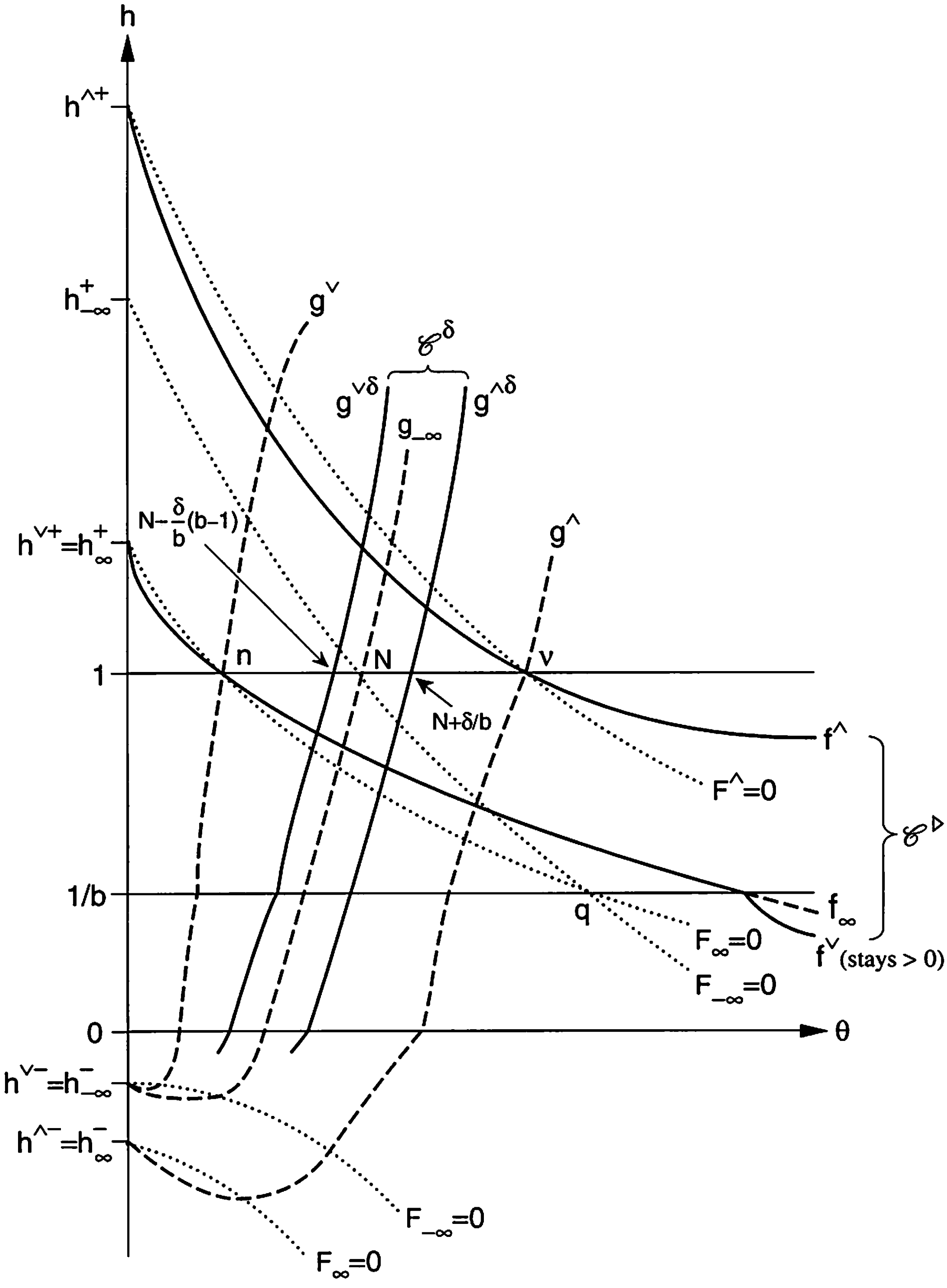

Figure 3(iii): 5–Parameter System, Type 1

$N > n > 0,\ b > 1,\ 0 < \psi'_o < m$

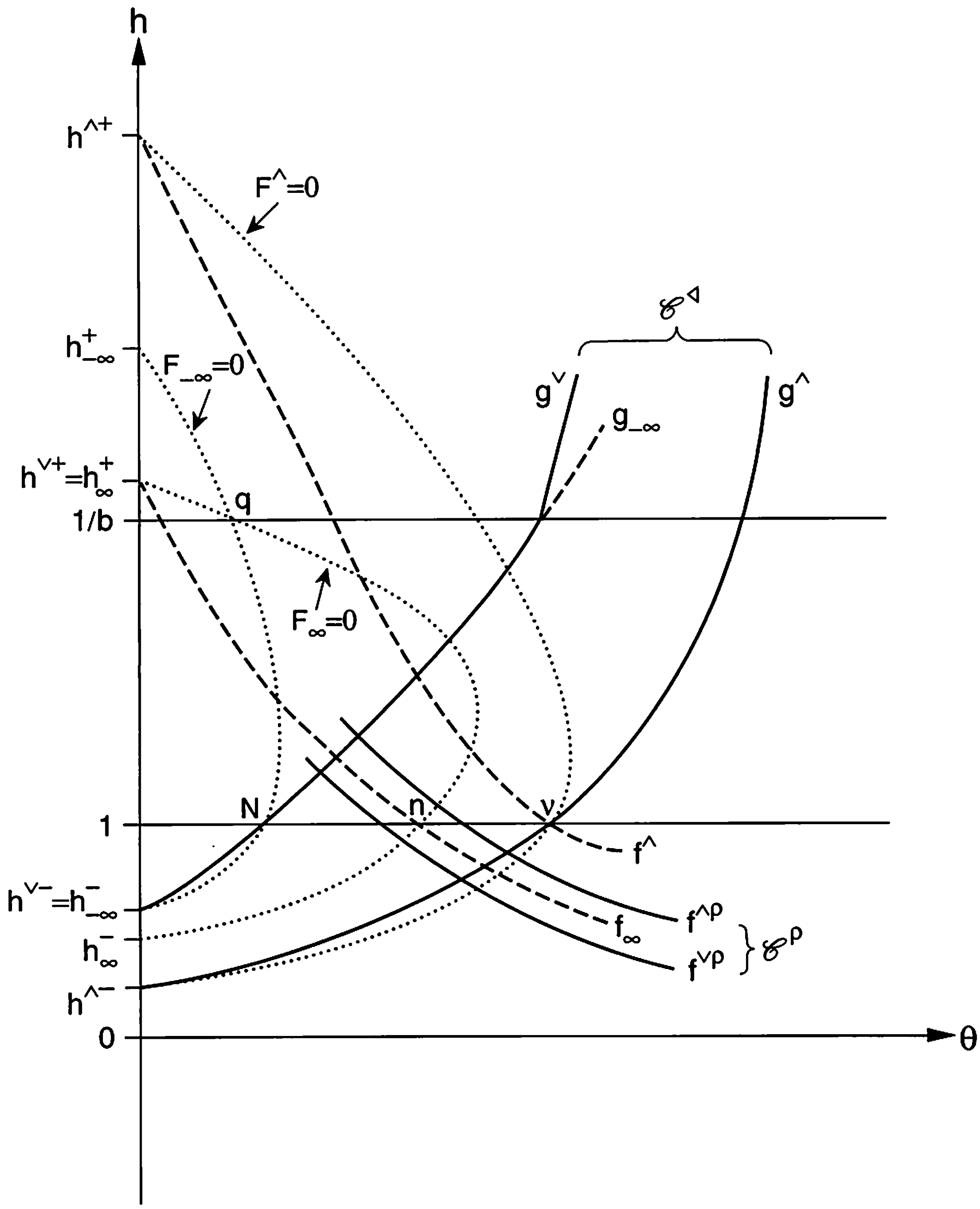

Figure 3(iv): 5–Parameter System, Type 1

$n > N > 0,\ b < 1,\ m < 0 < \psi_o'$

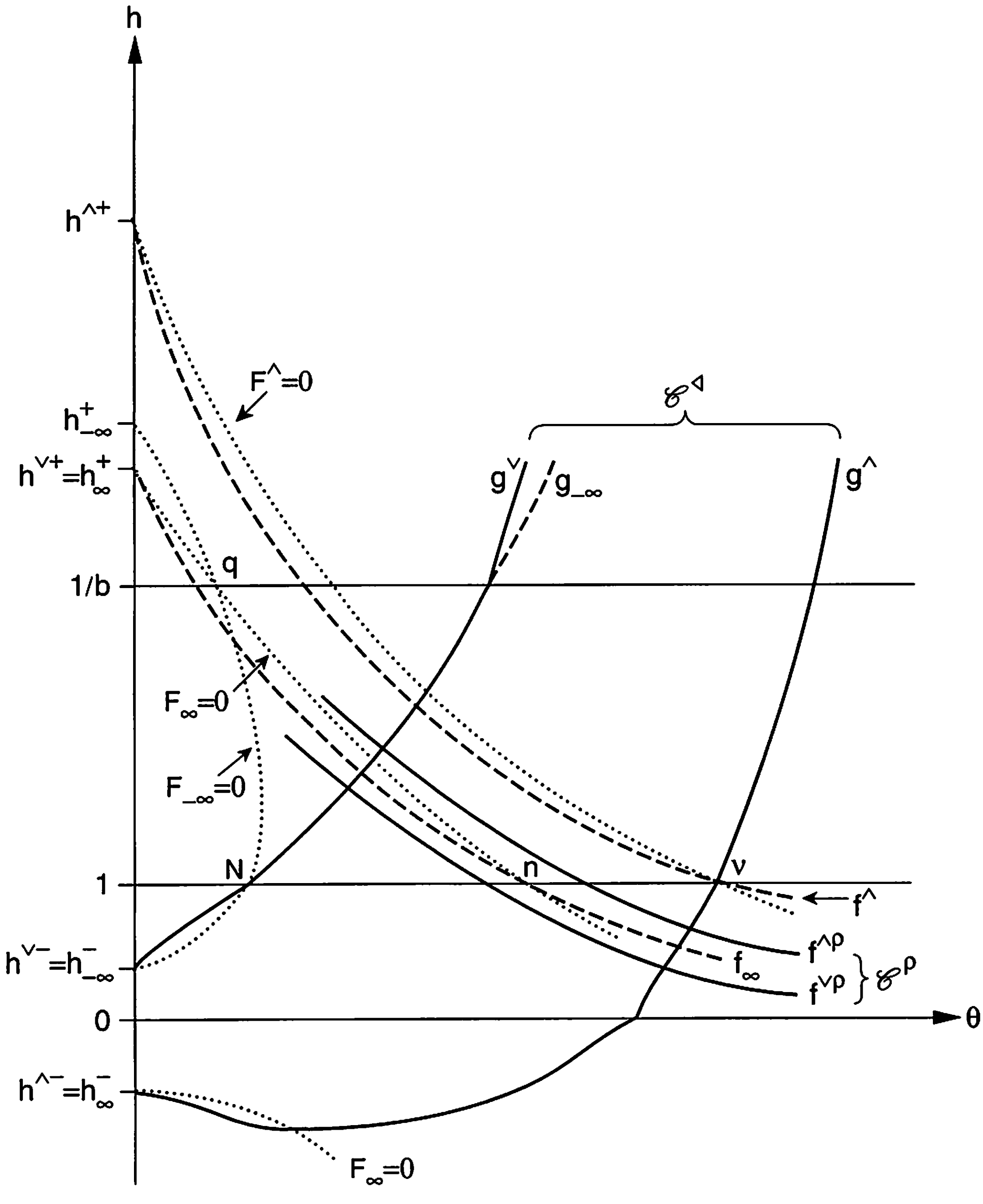

Figure 3(v): 5–Parameter System, Type 1

$n > N > 0,\ b < 1,\ 0 < m < \psi_o'$

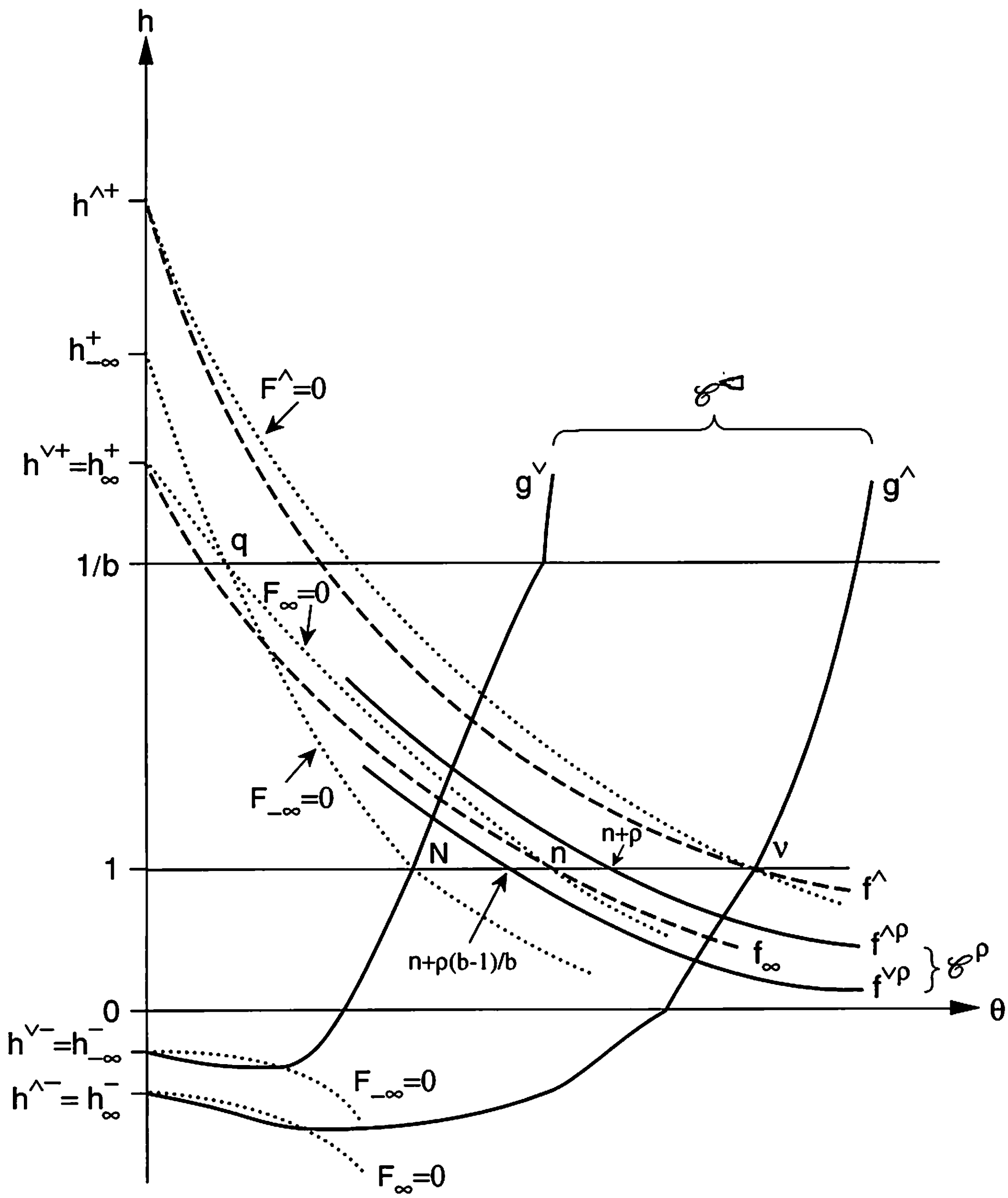

Figure 3(vi): 5–Parameter System, Type 1

$n > N > 0$, $b < 1$, $0 < \psi_o' < m$

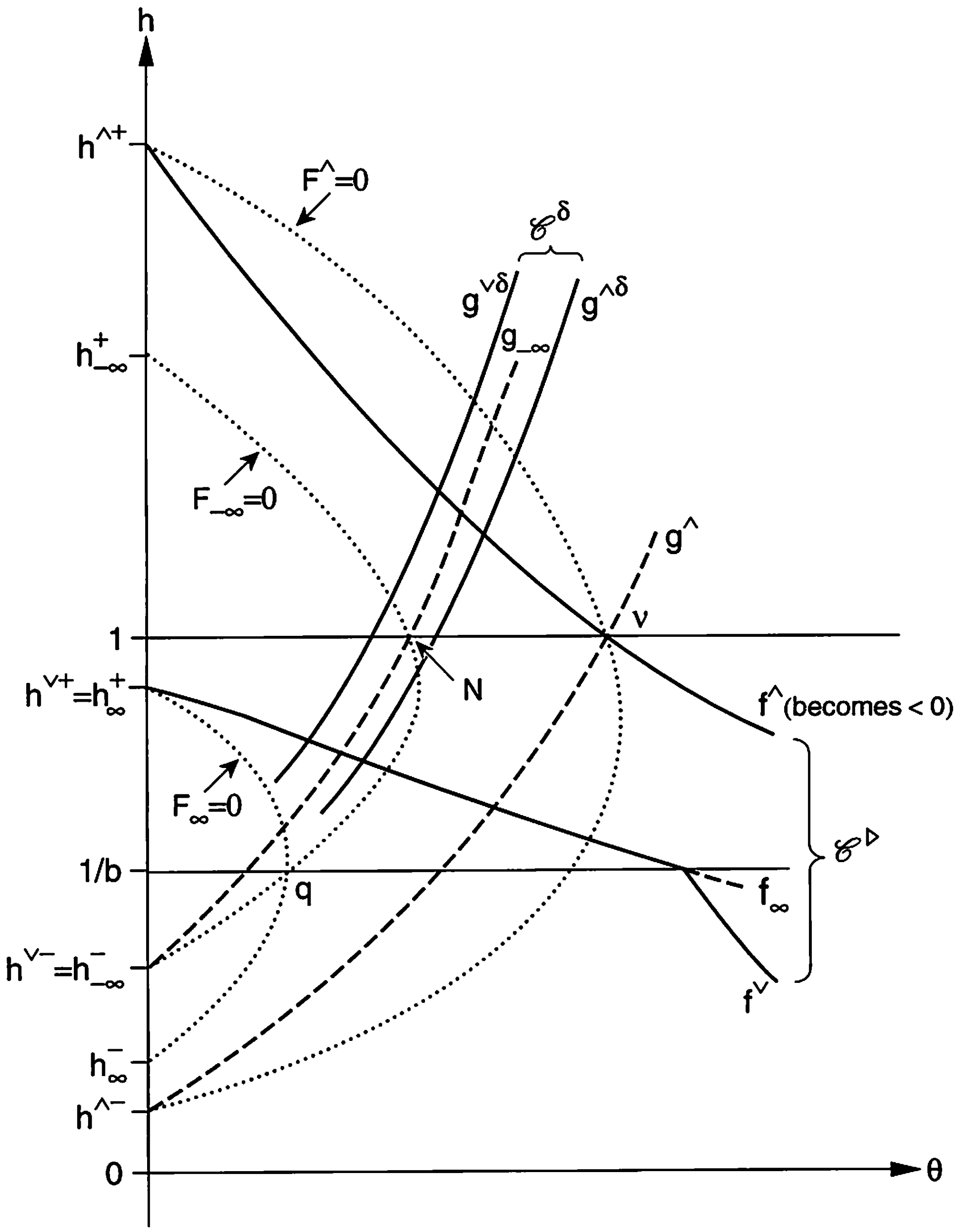

Figure 4(i): 5–Parameter System, Type 0

$N > 0 > n,\ q > 0,\ b > 1,\ m < 0 < \psi_o'$

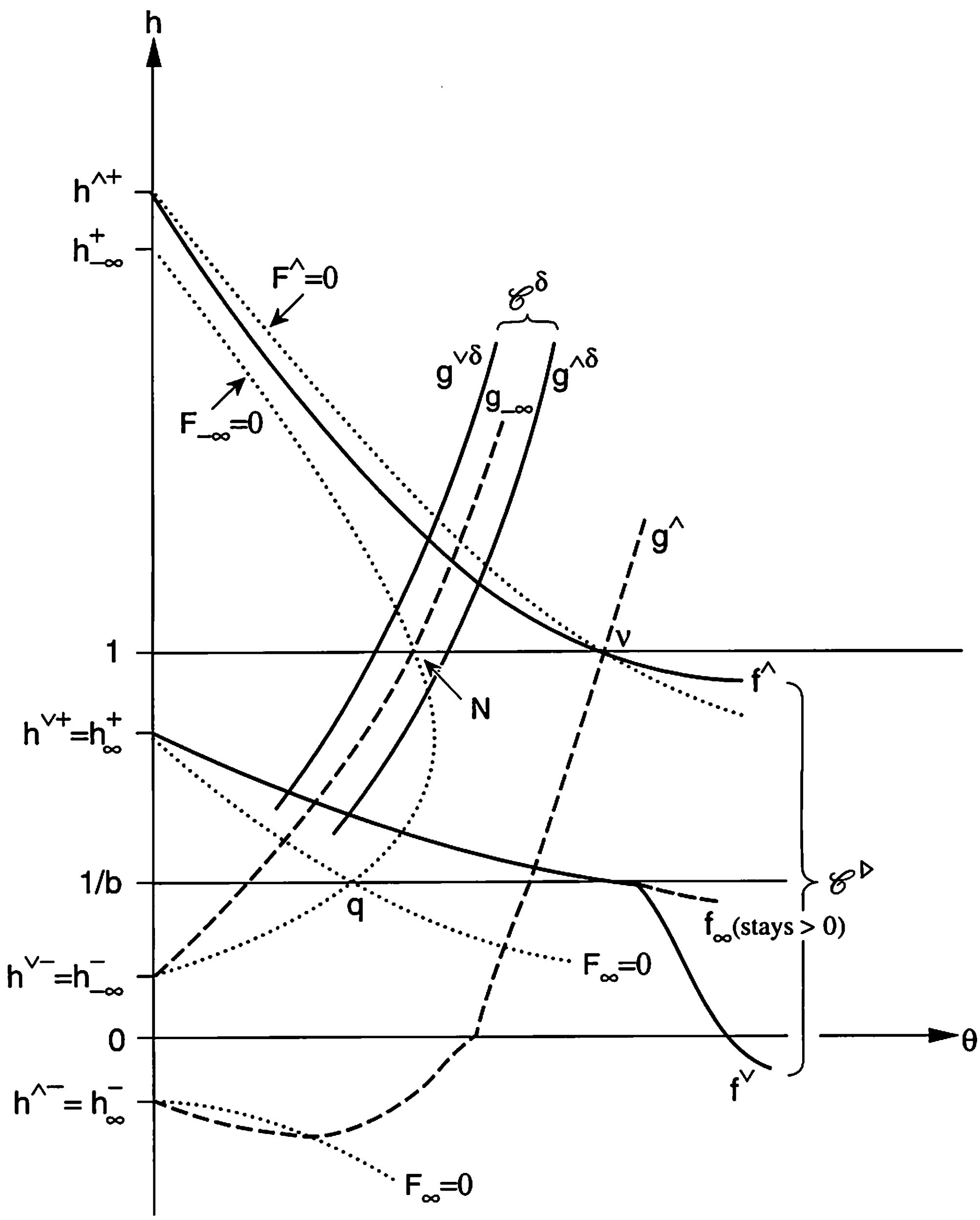

Figure 4(ii): 5−Parameter System, Type 0

$N > 0 > n,\ q > 0,\ b > 1,\ 0 < m < \psi_o'$

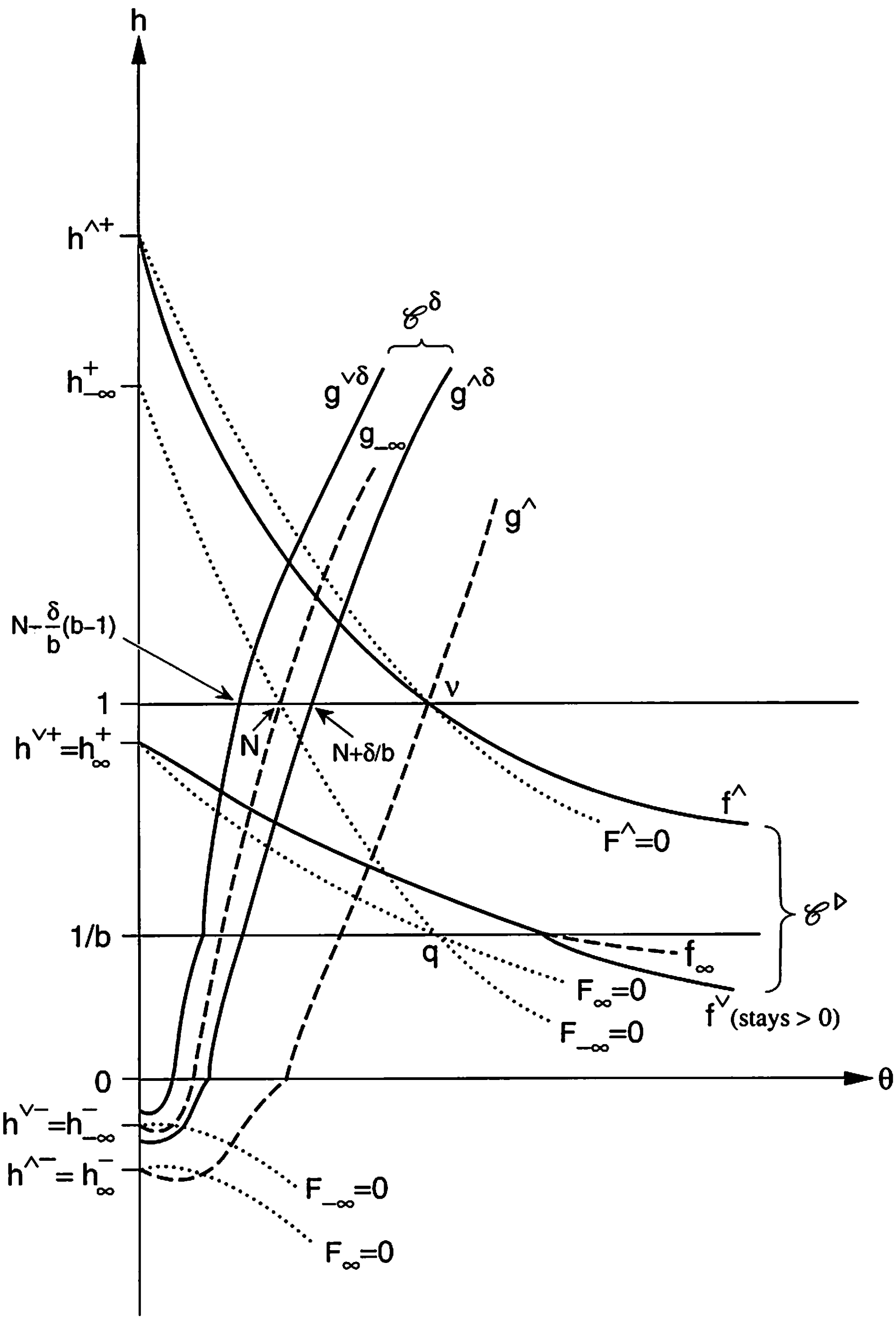

Figure 4(iii): 5–Parameter System, Type 0

$N > 0 > n,\ q > 0,\ b > 1,\ 0 < \psi_0' < m$

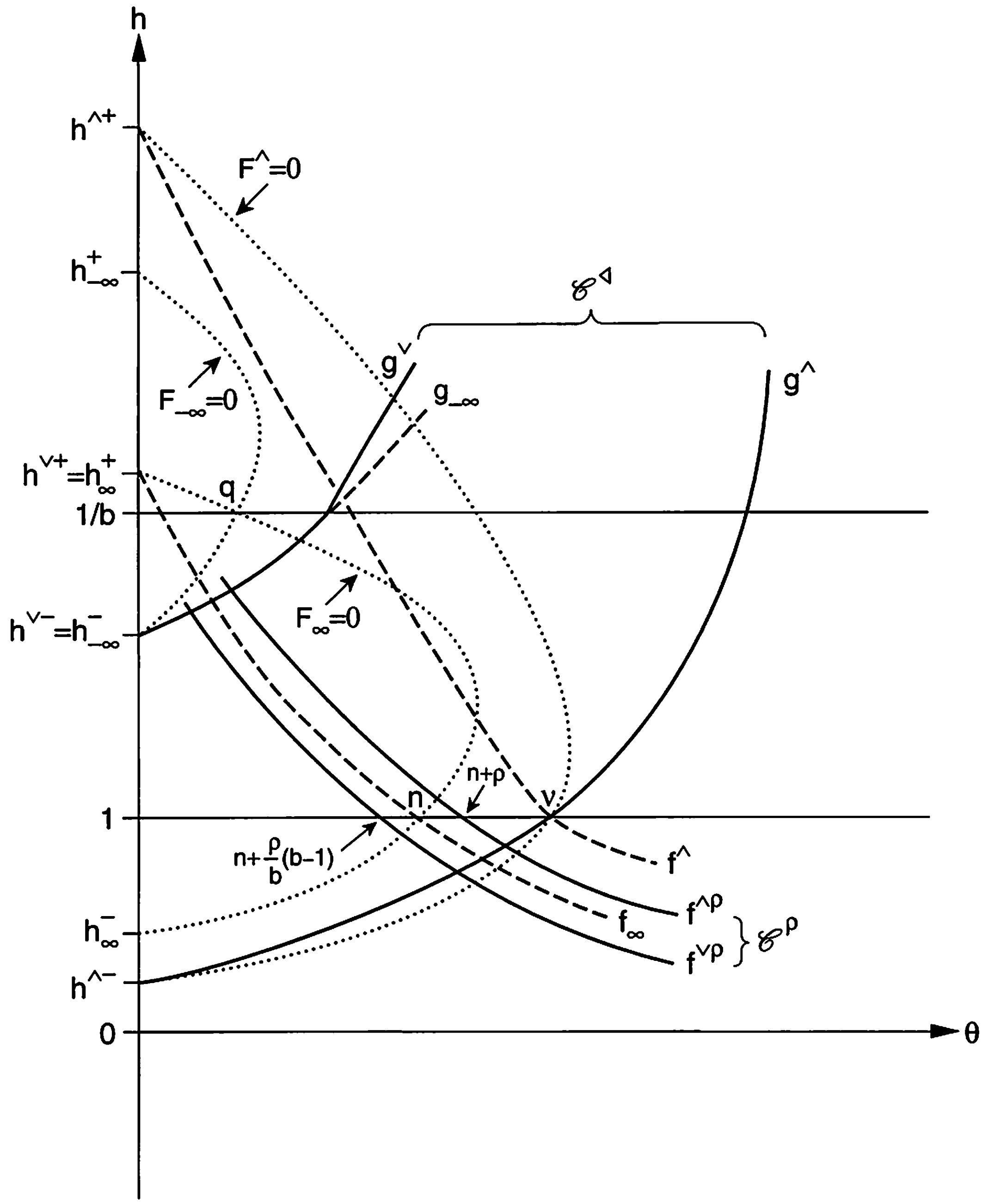

Figure 4(iv): 5–Parameter System, Type 0

$n > 0 > N,\ q > 0,\ b < 1,\ \ m < 0 < \psi_o'$

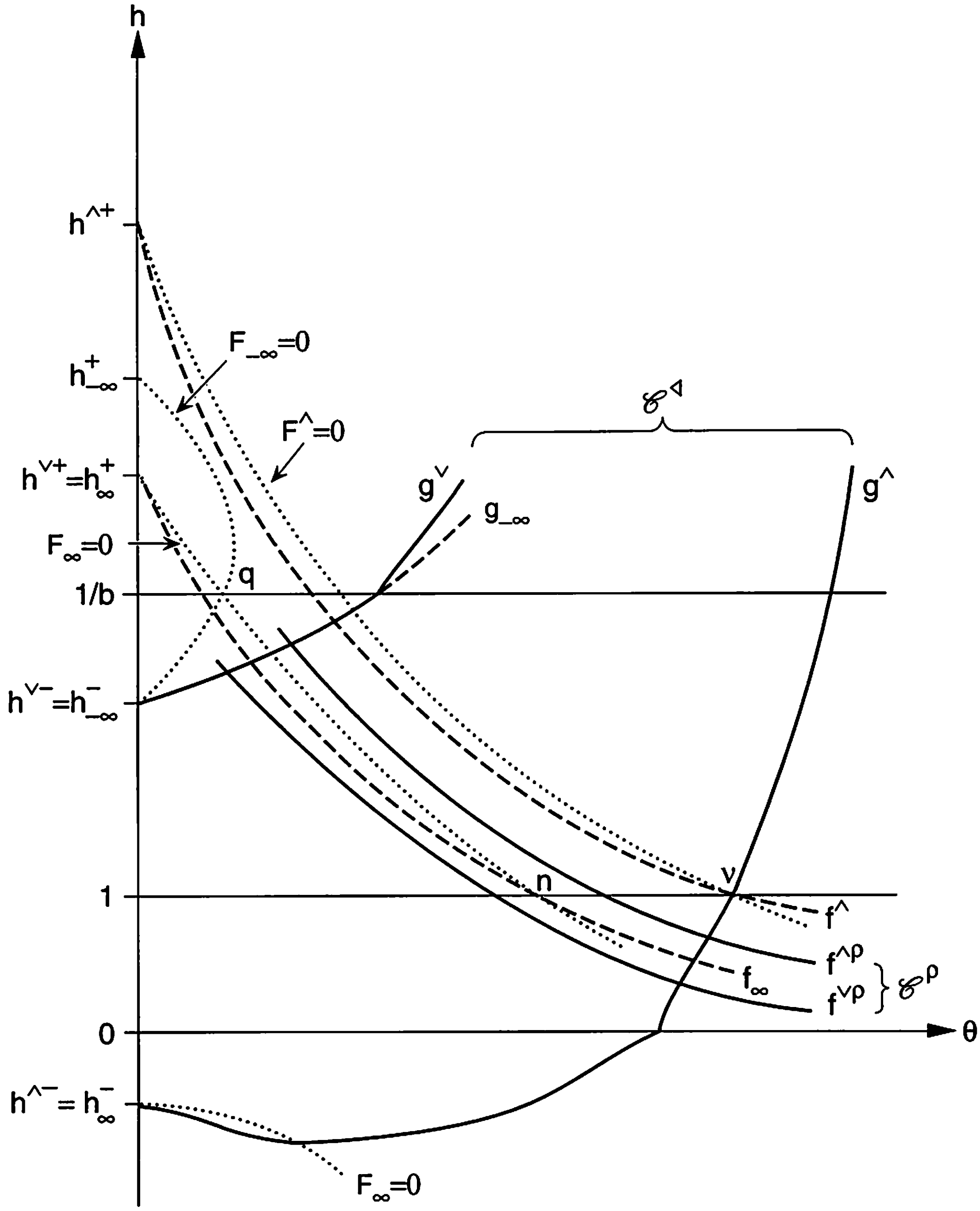

Figure 4(v): 5–Parameter System, Type 0

$n > 0 > N,\ q > 0,\ b < 1,\ 0 < m < \psi_o'$

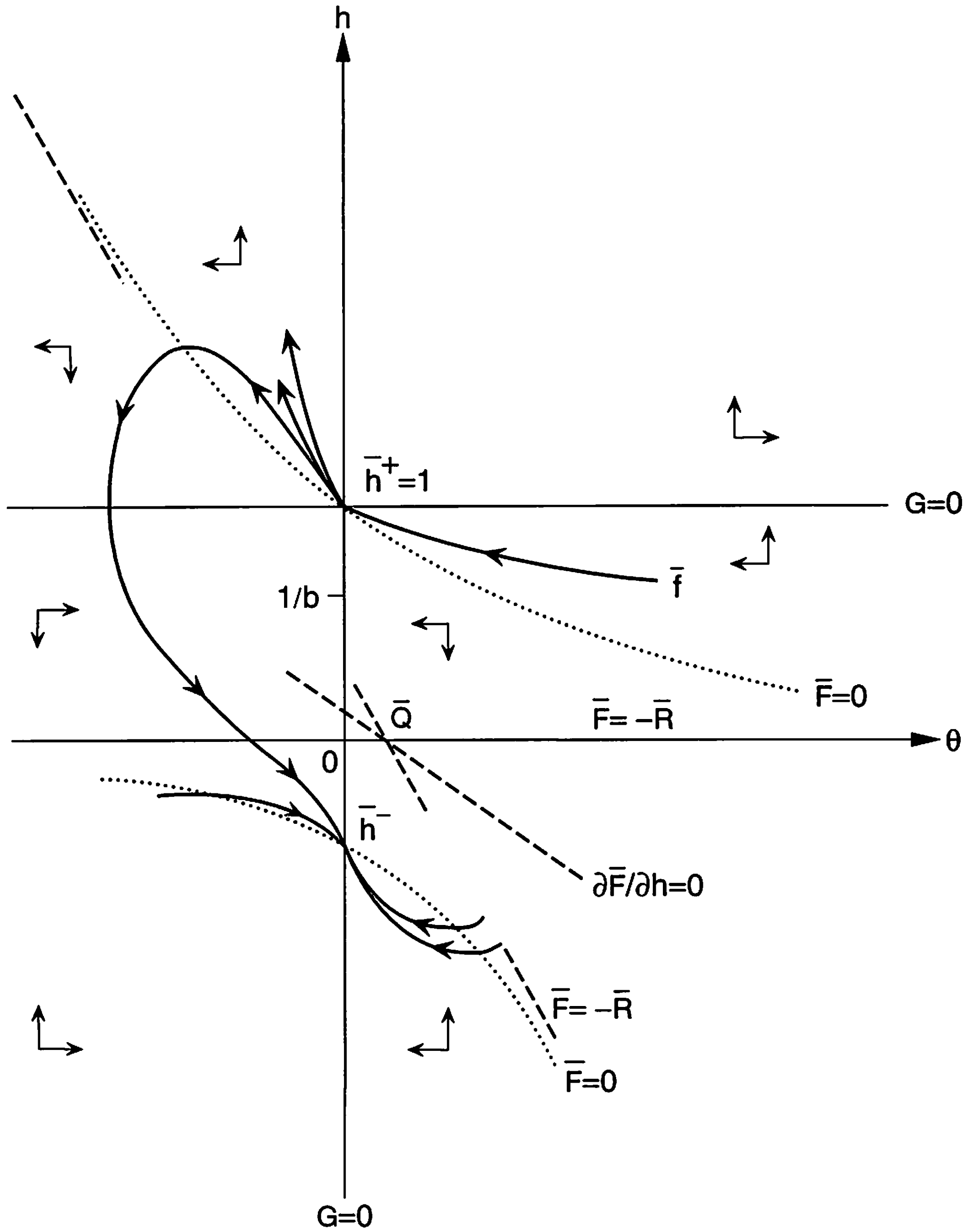

Figure 5: 3–Parameter System, Saddle–Node at (1,0)

$\bar{\theta}_1 = 0$, $b > 1$, $\bar{R} = 2\bar{m}/b\sigma^2 > 0$

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