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# Systemic Risk and the Dynamics of Temporary Financial Networks

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Keywords: systemic risk, counter-party risk, financial networks, supernetworks, tipping points, default cascades, basins of attraction, Pareto optimal stationary Markov equilibrium, first passage probabilities, hitting times, hitting time probabilities.

JEL Classification: C7

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# Systemic Risk and the Dynamics of Temporary Financial Networks


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# 1 Introduction

Since the financial crisis of 2007-2008, we have come to realize that in order to correctly assess the potential default risk present in any bilateral contractual relationship between two firms, it is essential that we have a clear picture of the network structure of each firm's bilateral contractual connections to other firms. These network connections are the channels or pathways over which, not only liquidity travels in moving through the network, but also over which default contagion travels in moving through the network. Moreover, in order to correctly assess the risk of a boarder network failure brought about by shocks to individual firms or groups of firms, we must know something about the strategic behavior of firms in responding to such shocks, as well as how the interplay between strategic behavior and network structure generate the dynamics which drive network formation. This "risk of a boarder network failure brought about by shocks to individual firms or groups of firms" is usually referred to as *systemic risk*. While we know systemic risk when we see it, surprisingly, we have no generally agreed upon formal definition of it - a fact pointed out by Glasserman and Young (2015) among others. This paper has two objectives: first, to provide a *formal definition of systemic risk* that is firmly grounded in the equilibrium dynamics of network formation; and second, to construct a discounted stochastic game (DSG) model of the *emergence of these equilibrium network dynamics* that fully takes into account the feedback between network structure, strategic behavior, and risk. Given the rules of network formation, the preferences of individuals over networks, and the environment of risk and uncertainty in which network formation takes place, it is the interactions between strategic behavior and network structure under conditions of risk that generate the equilibrium stochastic process of network formation. As a consequence, it is the strategic underpinnings of this process which must be understood in order to fully understand systemic risk (for a survey of the issues surrounding the notion of endogenous systemic risk, see Zigrand, 2014).

In addition to providing an alternative way of thinking about systemic risk - including a formal definition of systemic risk and a game-theoretic model of the equilibrium dynamics which generate systemic risk, we also provide a strategic approach to making global assessments of systemic risk in networks. Three key facts about equilibrium network dynamics make our approach to systemic risk potentially very useful. First, the equilibrium dynamics determined by our game-theoretic model of network formation uniquely partitions the entire state space of state-network pairs into a transient set together with a finite number of absorbing sets (i.e., basins of attraction) each consisting of state-network pairs which persist through time.[1] Moreover, these equilibrium state-network dynamics have the property that no matter where the process of network formation begins, in finite time with probability one, the process will enter one of these basins of attraction - and once there, will stay there. Thus, here we provide at least the beginnings of a strategically-based theory of long run prediction and risk assessment in financial networks. Second, each basin of attraction is homogenous with respect to its default characteristics - meaning that if a basin contains a state-network pair having a particular set of defaulted players, then all state-network pairs contained in this basin have the same set of defaulted players. Thus, despite there being uncountably many state-network pairs, there are only finitely

[1]Letting $\omega_t$ denote the state at time $t$, a state-network pair is given by

$$(\omega_t, G(\omega_t)).$$

$G(\omega_t)$ is the network resulting from the strategic connection choices proposed by players in state $\omega_t$. As the state process probabilistically moves through the state space, the corresponding, state-dependent network process moves through the space of networks. The Markov probabilities governing these movements are functions of the current state and the network connection proposals made by the players. We will soon fill in the details.

many network default configurations that can arise - and these default configurations and their probabilities of occurring can be computed. Third, each basin has a sphere of influence. These are subsets of transient state-network pairs which in a probabilistic sense belong to the basin in question - in that, if the process starts from a state-network pair contained in a particular basin's sphere of influence, then in finite time with probability one, the process will arrive at this basin and only this basin. Thus, if the basin in question is one with very bad default characteristics (i.e., each network in the basin has a large number of defaulted players), then each pathway from a state-network pair contained in this basin's sphere of influence to the basin itself can be thought of as a default cascade. Alternatively, if the basin in question is one with very good default characteristics (i.e., each network in the basin has a very few or no defaulted player), then each pathway from a state-network pair in this basin's sphere of influence to the basin itself can be thought of as a default extinguishing path - or as a path along which the equilibrium network dynamics are such that default is self-limiting. Moreover, at the boundary of these spheres of influence are the tipping point state-network pairs.

A useful visualization devise for understanding of our approach to systemic risk is to think of the equilibrium (state) dynamics as being represented by a *supernetwork* where the nodes are the possible states and the directed arcs pointing from one state to another are labeled by the equilibrium transition probabilities of the network moving from one state to another. These equilibrium transition probabilities are a function of the profile of players' network formation strategies. By way of a useful analogy, if we then think of this supernetwork as representing the "transportation network" over which the network will travel in moving from one state to another, we can then compute the probabilities that the current network, departing from its current state (perhaps one with no defaulted players), arrives at any other state or set of states (having defaulted players) at or before a particular time. With our transportation analogy in mind, we are led *to define the systemic risk of the current network as the first passage probability to some future state or set of states identified as having a particular subset of defaulted players*. The "time" with respect to which our first passage probabilities are computed can be given by a particular time point, by an interval of time, or by all finite times. We then have for each possible current network in a particular state a schedule of systemic risk measures indexed by times and states (and subsets of states). Under our approach, systemic risk is better thought of as *a transportation (to default) schedule*, relevant to the current network in its current state (a state from which the network will depart), giving the probabilistic arrival times of the network at various failed states or subsets of failed states (where failure is characterized by a particular subset of defaulted players *chosen by the observer who is seeking to measure systemic risk*). Under our definition of systemic risk (as a first passage probability), systemic risk is inextricably linked to the equilibrium network dynamics determined by the interplay between strategic behavior, network structure, and risk. Moreover, by its very definition, our notion of systemic risk takes into account the timing and severity of the risks being measured.

The presence of finitely many basins of attraction, each consisting of states having a particular subset of defaulted players, together with the fact that the current network - no matter what its current state - will arrive at one of these basins (i.e., will arrive at some state contained in one of these basins) in finite time with probability one, has major implications for our understanding of how best to measure and control systemic risk. Moreover, the presence of a unique set of spheres of influence, serving as an early warning system for impending defaults, enhances our ability to guide the network formation process toward less systemically risky states by providing a set of navigation beacons (keeping in mind our transportation analogy). The big picture take away from our approach to systemic risk is that what really matters in assessing the potential severity of the systemic risk of a network is the distribution of the defaulted players across finitely many basins of attraction. In fact, if in the collection of basins of attraction there is a basin with no defaulted players, while all other basins contain only a few defaulted players, then we could describe the *network formation process as being resilient because with positive probability it endogenously limits default.* This raises an interesting question: is there a way, using smartly designed policies, that we can guarantee that the equilibrium stochastic process of network formation is resilient? While we do not take up a detailed analysis of this question here, our game-theoretic model and our definition of systemic risk provide the tools with which such an investigation can be carried out.

Here, in developing our definition of systemic risk, we will focus on short-term lending and investment networks - saving for future work the much more difficult problem of analyzing of the interconnections of systemic risk (as defined here) and the maturity structure of lending and investments. We will refer to our short-term lending and investment networks as *temporary financial networks.* In the discounted stochastic game model of temporary financial network formation constructed here, each of $n$ players forms two networks: (i) one consisting of short-term borrowing or lending connections with the other players, and (ii) one consisting of investment connections with some subset of $m$ possible (perfectly divisible) risky investment projects. Players borrow or lend short term in order to adjust their levels of investable funds available for the risky projects. The formation of the borrowing and lending network takes place in two steps. First, each player, $i$, proposes a profile of borrowing or lending contracts to the other players. Each such proposed contract, for example one from player $i$ to player $j$, is specified by a proposed amount $l^0_{ij}$ to be borrowed ($< 0$) or lent ($> 0$) at the beginning of the period and a proposed amount $l^1_{ij}$ to be repaid at the end of the period. Player $i's$ contract proposal, $(l^0_{ij}, l^1_{ij})$, to $j$ becomes a real connection between $i$ and $j$ in the borrowing and lending network if - given $i's$ proposal, $(l^0_{ij}, l^1_{ij})$ - $j's$ counter-proposal, $(l^0_{ji}, l^1_{ji})$, to $i$ is matching - that is, if $l^t_{ij} + l^t_{ji} = 0$ for $t = 0$ and 1. If player pair $ij$'s contract proposals fail to match, then a matching is reached through bargaining between players $i$ and $j$. Here, rather than model this bargaining process explicitly, we instead assume that there is a matching function which incentivizies players to reach a matching in borrowing or lending proposals. Once players have reached borrowing and lending matches (thereby determining their borrowing-lending network and their levels of investable funds), players choose an allocation of their investable funds across the $m$ risky investment projects. If as a result of prior investment, borrowing and lending activity, a player begins the period with insufficient cash, then during the coming period, the player is allowed to try to borrow sufficient funds to continue (i.e., to bring his investable funds level to a positive amount). Failing this, at the beginning of the next period the player becomes a permanent member of the set of defaulted players - and remains inactive in perpetuity. In order to take into account the unintended network-wide, negative cash flow consequences of a player's (or players') default, we adjust players' debt repayments to reflect the actual debt repayments players are able to make after a default. Here, using the Eisenberg-Noe (2001) approach, we obtain a stationary default adjustment function which allows us to compute the equilibrium default adjusted repayments. Moreover, using our stationary default adjusted repayments function, we are able to compute the equilibrium, default adjusted, short-term lending rate process.

We will proceed according to the following outline:

2. Temporary Financial Networks

   2.1 The State Space

   2.2 Short-Term Lending Networks

# 2 Temporary Financial Networks

We consider the problem faced by a set of players who at the beginning of each period $[t, t+1]$ (at time $t$), after observing the state, $\omega_t := (C_t, D_t, s_t)$, and therefore after observing the cash flow levels, $C_t$, available for the coming period, form a financial network of investment connections, as well as borrowing-lending-repayment connections. The payoffs generated by forming such networks are risky. This is because these payoffs are largely determined - not by the current state - but by the coming state. As to which coming state will occur, players only know the conditional probabilities - hence the risk. Moreover, the conditional probabilities governing the occurrence of upcoming states are largely determined by the strategic interactions of the players in forming networks. Our objective here is to provide a game-theoretic model of these strategic interactions, and more importantly, to provide an understanding of how these interactions determine the equilibrium state dynamics and therefore the risks inherent in these dynamics.

We will assume that the set of players is given by $N := \{1, 2, \ldots, n\}$, with typical elements $i$ and $j$.

## 2.1 The State Space

The states about which players are uncertainty reside in the set $\Omega$ with typical element, $\omega = (C, D, s)$, consisting of the $n$-tuple of player cash flows, $C := (C^1, \ldots, C^n) \in M^n$, where $M := [-H, H]$, $H > 0$, the subset of defaulted players, $D \in 2^N$, where $2^N$ is the collection of all subsets of $N$ including the empty set, and the state of the real economy $s \in S$, where $S$ is a complete, separable metric space with metric $\rho_S$. Thus, the set of states is given by

$$\Omega := M^n \times 2^N \times S.$$

We will equip $\Omega$ with the product $\sigma$-field,

$$B_\Omega := B_{M^n} \times 2^{2^N} \times B_S,$$

where $B_{M^n}$ is the Borel product $\sigma$-field in $M^n$, $2^{2^N}$ is the set of all subsets of $2^N$, and $B_S$ is the Borel $\sigma$-field in $S$. Finally, we will equip $\Omega := M^n \times 2^N \times S$ with the product probability measure,

$$\mu := \lambda \times \eta \times \nu.$$

Thus, the state space is given by the probability space

$$(\Omega, B_\Omega, \mu) = (\underbrace{M^n \times 2^N \times S}_{\text{states}}, \underbrace{B_{M^n} \times 2^{2^N} \times B_S}_{\text{events}}, \underbrace{\lambda \times \eta \times \nu}_{\text{probabilities}}). \tag{1}$$

If at time $t$ the state is

$$\omega_t = (C_t, D_t, s_t) \in \Omega = M^n \times 2^N \times S. \tag{2}$$

then player $i's$ available cash for the coming period, $[t, t+1]$, is

$$C^i(\omega_t) := proj_{M^i}(\omega_t) = C^i_t \in M^i, \tag{3}$$

the set of players who are in default during the coming period, $[t, t+1]$, is

$$D(\omega_t) := proj_{2^N}(\omega_t) = D_t \in 2^N, \tag{4}$$

and the state of the real economy during the coming period, $[t, t+1]$, is

$$s(\omega_t) := proj_S(\omega_t) = s_t \in S. \tag{5}$$

(Here $proj_{(\cdot)}(\cdot)$ is the usual projection function).

## 2.2 Short-Term Lending Networks

Each non-defaulted player, $i \in N \backslash D_t$, at the beginning of period $[t, t+1]$, $t \in T := \{0, 1, 2, \ldots\}$, after observing the state $\omega_t$ and learning his cash flow, $C^i$, *proposes* two new $n$-tuples of borrowing-lending-repayment amounts, $l^i := (l^{0i}, l^{1i})$ - in this way, the player can augment his current cash flow to obtain the desired level of investable funds. This proposed pair of $n$-tuples potentially represents player $i$'s part of the loanable funds network for the coming period (i.e., $[t, t+1]$). The first $n$-tuple,

$$l^{0i} := (l^0_{i1}, l^0_{i2}, \ldots, l^0_{in}) \in M^n, \tag{6}$$

is player $i$'s borrowing and lending proposals to all other players. The second $n$-tuple,

$$l^{1i} := (l^1_{i1}, l^1_{i2}, \ldots, l^1_{in}) \in M^n, \tag{7}$$

is player $i$'s repayment proposals to all other players. Recall, $M$ is the closed bounded interval $[-H, H]$, $H > 0$, with $M_- := [-H, 0]$ and $M_{++} := (0, H]$, and

$$M^n := \underbrace{M \times \cdots \times M}_{\text{n times}}.$$

The the $j^{th}$ components, $(l^0_{ij}, l^1_{ij})$, of each of the two $n$-tuples, $(l^{0i}, l^{1i}) \in M^n \times M^n$, specifies the bilateral borrowing or lending contract proposed by player $i$ to player $j$. If the $j^{th}$ components, $(l^0_{ij}, l^1_{ij})$ of each of the two $n$-tuples, $(l^{0i}, l^{1i})$, are such that

$$l^0_{ij} < 0 \text{ and } l^1_{ij} > 0, \tag{8}$$

then player $i$ is proposing to borrow an amount $-l^0_{ij} > 0$ from player $j$ and pay back an amount $l^1_{ij} > 0$ to player $j$ at the end of the period. If the $j^{th}$ components, $(l^0_{ij}, l^1_{ij})$ of each of the two $n$-tuples, $(l^{0i}, l^{1i})$, are such that

$$l^0_{ij} > 0 \text{ and } l^1_{ij} < 0, \tag{9}$$

then player $i$ is proposing to lend an amount $l^0_{ij} > 0$ to player $j$ at the beginning of the period and to be paid back an amount $-l^1_{ij} > 0$ by player $j$ at the end of the period.

### 2.2.1 Borrowing, Lending, and Default

Given player $i$'s cash flow, $C^i_t$, at time point $t$ (the beginning of period $[t, t+1]$), player $i$'s borrowing-lending-repayment proposals, $(l^{0i}, l^{1i}) \in M^n \times M^n$, must be such that if $C^i \leq 0$, then $l^{0i} := (l^0_{i1}, l^0_{i2}, \ldots, l^0_{in}) \in M^n_-$ (i.e., $l^0_{ij} \in M_-$ for all $j \in N$). Thus, if at time point $t$ player $i's$ cash flow level is nonpositive, then player $i$ is constrained to make only borrowing proposals (i.e., player $i$ is constrained to try to augment his cash flow enough via borrowing so that his level of investable funds is positive) - but even under this constraint there are three possible proposal decisions player $i$ can make: (1) player $i$ can choose to permanently enter the set of defaulted players at $t+1$ by choosing at $t$ to make no borrowing proposal (by choosing $l^{0i} = 0$), (2) player $i$ can choose to make a borrowing proposal, $l^0_{ij} \leq 0$ for all $j \in N$, but one that is insufficient in that $C^i - \sum_{j \in N} l^0_{ij} := C^i - \langle l^{0i}, e \rangle \leq 0$, or (3) player $i$ can choose to make a borrowing proposal, $l^0_{ij} \leq 0$ for all $j \in N$, such that $C^i - \langle l^{0i}, e \rangle > 0$. The first two decisions will result in player $i$ qualifying for permanent membership in the set of defaulted players starting at time point $t+1$. Thus, if for player $i$, $C^i_t \leq 0$ and player chooses a network proposal, $(\pi^i(\omega_t), l^{0i}(\omega_t), l^{1i}(\omega_t)) \in \Delta(Q) \times M^n_- \times M^n$, at time point $t$ in state $\omega_t$ such that $C_t^i - \langle l^{0i}(\omega_t), e \rangle \leq 0$, then in the coming state, $\omega_{t+1} := (C_{t+1}, D_{t+1}, s_{t+1})$, at time point $t+1$, player will be a new member of the set of defaulted players, $D_{t+1}$ (note that $i \notin D_t$) - and will remain a member for all $t' \geq t+1$. These observations provide us with a way to identify those players who are prime candidates for permanent membership in the set of defaulted players. In particular, in state $\omega$, the set of players who qualify for permanent membership in the set of defaulted players is given by,

$$N_f(\omega) := \left\{ i \in N : C^i(\omega) := C^i \leq 0 \text{ and } C^i - \langle l^{0i}(\omega), e \rangle \leq 0 \right\}. \tag{10}$$

Conversely, for player $i$ the set of states in which player $i$ qualifies for permanent membership in the set of defaulted players is

$$\Omega_f(i) := \left\{ \omega \in \Omega : C^i(\omega) := C^i \leq 0 \text{ and } C^i - \langle l^{0i}(\omega), e \rangle \leq 0 \right\}. \tag{11}$$

Thus, if at time $t$ in state $\omega_t$, $i \in N_f(\omega_t)$, then at time $t+1$, $i \in D_{t+1}$. Moreover, if $t$ is the first time that $\omega_t \in \Omega_f(i)$, then at time $t+1$, $i \in D_{t+1}$ and for all future time points, $t' \geq t+1$, $i \in D_{t'}$. We will return to our discussion of default below when we discuss the equilibrium state process and introduce our discounted stochastic game model of financial network formation - the game from which these dynamics emerge.

At time point $t$, each player $i \in N \backslash D_t$ faces a borrowing-lending constraint correspondence, $C \longrightarrow \mathbb{B}(C)$, for borrowing and lending proposals defined on the player $i$'s cash flow levels, $C^i \in M$. For each possible level of cash flows, $C^i \in M$, the set of feasible borrowing and lending proposals, $\mathbb{B}(C^i)$, is given by

$$\mathbb{B}(C^i) := M_-^n I_{[C^i \leq 0]}(C^i) + M^n I_{[C^i > 0]}(C^i), \tag{12}$$

where $I_{[C^i \leq 0]}(\cdot)$ and $I_{[C^i > 0]}(\cdot)$ are indicator functions for the intervals $[C^i \leq 0]$ and $[C^i > 0]$, respectively.

Viewing each player's borrowing-lending-repayment proposal, $(l^{0i}, l^{1i})$, as row vectors, we can use these row vectors, one pair from each player, to form two $n \times n$ matrices, $L^0$ and $L^1$, with the $n$-tuple $l^{ti}$ forming the $i^{th}$ row of the matrix $L^t$, $t = 0, 1$. Together the matrices,

$$(L^0, L^1) := (l^{0i}, l^{1i})_{i \in N}, \tag{13}$$

with feasible rows $(l^{0i}, l^{1i}) \in \mathbb{B}(C^i) \times M^n$, each represent a player's loanable funds network proposal. In matrix form, we indicate that a loanable funds network proposal profile,

$$(L^0, L^1) := (l^{0i}, l^{1i})_{i \in N} := ([l^0_{ij}]_{ij \in N^2}, [l^1_{ij}]_{ij \in N^2}),$$

is feasible by writing $(L^0, L^1) \in \mathbb{A}(C)$, where

$$\mathbb{A}(C) := \left[ \mathbb{B}(C^1) \times \cdots \times \mathbb{B}(C^n) \right] \times \left[ \mathbb{G}^1 \times \cdots \times \mathbb{G}^n \right] := \mathbb{B}(C) \times \mathbb{G}, \tag{14}$$

where

$$\mathbb{G}^i := \{ l^{1i} = (l^1_{i1}, \ldots, l^1_{in}) \in R^n : i \in N, \, l^{1i} \in M^n \}. \tag{15}$$

### 2.2.2 Matching

In order for a feasible proposed network to become a network, it must be bilaterally consistent or matching. A feasible loanable funds network proposal by players,

$$(L^0, L^1) := ([l^0_{ij}]_{ij \in N^2}, [l^1_{ij}]_{ij \in N^2}) \in \mathbb{A}(C), \tag{16}$$

is matching if for each player pair, $ij$,

$$l^t_{ij} + l^t_{ji} = 0 \text{ for } t = 0, 1.$$

Thus, if for player pair, $ij$, the borrowing or lending proposed by player $i$ to player $j$ is consistent with that proposed by player $j$ to player $i$, then $L^0 \in \mathbb{B}(C)$ is matching, and if for player pair $ij$, the payments or repayments proposed by player $i$ to player $j$ is consistent with that proposed by player $j$ to player $i$, then $L^1$ is matching. In matrix form (with rows representing player proposals), the set of matching matrices is given by

$$\mathbb{M} := \{L := [l_{ij}]_{ij \in N^2} \in \mathbb{G} : \forall\, ij,\, l_{ij} + l_{ji} = 0\}. \tag{17}$$

Note that $L^t \in \mathbb{M}$ ($t = 0$ or 1) if and only if $L^t = -[L^t]^T$ (i.e., $L^t$ is equal to the negative of its transpose).

[**A-1**] *(How Proposed Networks Become Networks)*
*If the proposed loanable funds network, $(L^0, L^1)$, is feasible and matching, that is, if proposal $(L^0, L^1)$ is contained in $\mathbb{A}(C)$ and if $L^0 \in \mathbb{M}$ and $L^1 \in \mathbb{M}$, then $(L^0, L^1)$ moves from being the proposed loanable funds network to being the loanable funds network in force. The collection of all feasible, matching loanable funds networks is given by*

$$\mathbb{AM}(C) := (\mathbb{B}(C) \cap \mathbb{M}) \times \mathbb{M} \tag{18}$$

Note that the set of feasible network proposals, $\mathbb{A}(C) := \mathbb{B}(C) \times \mathbb{G}$, is a compact convex subset of $\mathbb{G} \times \mathbb{G}$. More importantly, note that the set of feasible matching network proposals $\mathbb{AM}(C) := (\mathbb{B}(C) \cap \mathbb{M}) \times \mathbb{M}$ is a closed convex subset of $\mathbb{A}(C)$. If a feasible network proposal, $L := (L^0, L^1) \in \mathbb{A}(C)$, is not matching, so that $L \notin \mathbb{AM}(C)$, then the move from the proposed network $L \in \mathbb{A}(C)$ to a feasible, matching network $L' \in \mathbb{AM}(C)$ is brought about by discussion, bargaining, and compromise. Here we will not explicitly model the bargaining process. Instead, we will assume that the outcome of the bargaining process is given by a state-dependent matching function. We will return to our matching outcome function once we have introduced the notion of an investments network.

## 2.3 Short-Term Investments Networks

Each player's cash inflow at the end of each period is generated by the returns on a portfolio of risky project investments made at the beginning of the period. Thus, at the beginning of each period, each player chooses an allocation of his investable funds, $max\left\{C^i - \left\langle l^{0i}(\omega), e\right\rangle, 0\right\}$, across a finite set of risky projects. We have the the following notation and assumptions:

[**A-2**] *(Short-Term Investments Networks)*
$Q = \{0, 1, 2, \ldots, m\}$ *the index set for investment projects.*

$\Delta(Q)$ = *the set of all probability measures,* $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$ *on* $Q$*, i.e.,*
$\pi := (\pi_q)_{q \in Q}$ *such that* $\forall\, q \in Q$, $\pi_q \in [0, 1]$ *and* $\sum_{q \in Q} \pi_q = 1$.
$r_q$ = *the random one-period return per dollar invested in project*
$q \in Q := \{0, 1, 2, \ldots, m\}$.
$C^i$ = *the amount of cash available to player* $i$ *(without borrowing).*
$\pi^i$ = *the* $i^{th}$ *player's portfolio weights,* $\pi^i := (\pi_{iq})_{q \in Q} \in \Delta(Q)$ *where* $\pi_{iq}$ *is the fraction of investable funds allocated by player* $i$ *to project* $q$.
$r = (r_0, r_1, \ldots, r_m) \in \mathcal{R}$ *is the random vector of risky project returns per dollar invested.*

Viewing each player's portfolio weights,

$$\pi^i := (\pi_{i0}, \pi_{i1}, \ldots, \pi_{im}) \in \Delta(Q), \tag{19}$$

as a row vector, we can put together these row vectors to form an $n \times n$ (stochastic) matrix $\Pi$, given by

$$\Pi := \begin{pmatrix} \pi_{10} & \pi_{11} & \cdots & \pi_{1k} & \cdots & \pi_{1m} \\ \vdots & \vdots & & \vdots & & \vdots \\ \pi_{i0} & \pi_{i1} & \cdots & \pi_{ik} & \cdots & \pi_{im} \\ \vdots & \vdots & & \vdots & & \vdots \\ \pi_{n0} & \pi_{n1} & \cdots & \pi_{nk} & \cdots & \pi_{nm} \end{pmatrix} \tag{20}$$

with row and column representations given by

$$\Pi := \underbrace{\begin{pmatrix} \cdots & \pi^1 & \cdots \\ & \vdots & \\ \cdots & \pi^i & \cdots \\ & \vdots & \\ \cdots & \pi^n & \cdots \end{pmatrix}}_{\text{row representation}} := \underbrace{\begin{pmatrix} \vdots & \vdots & & \vdots & & \vdots \\ \pi_0 & \pi_1 & \cdots & \pi_q & \cdots & \pi_m \\ \vdots & \vdots & & \vdots & & \vdots \end{pmatrix}}_{\text{column representation}}. \tag{21}$$

The $i^{th}$ row, $\pi^i$, of matrix $\Pi$ is player $i's$ portfolio allocation. The $q^{th}$ column, $\pi_q$, is the allocation across players of the return generated by project $q$. Given the vector, $C := (C^1, \dots, C^n)$, of players' cash flows, the vector of project returns, $r$, and the matrix of portfolio allocations, we can compute the end of period cash flow to each player.[2] In particular, we have

$$\begin{pmatrix} \langle r, \pi^1 \rangle C^1 \\ \vdots \\ \langle r, \pi^i \rangle C^i \\ \vdots \\ \langle r, \pi^n \rangle C^n \end{pmatrix} = \begin{pmatrix} C^1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & C^i & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & C^n \end{pmatrix} \begin{pmatrix} \cdots & \pi^1 & \cdots \\ & \vdots & \\ \cdots & \pi^i & \cdots \\ & & \\ \cdots & \pi^n & \cdots \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_m \end{pmatrix}, \tag{22}$$

where $\langle r, \pi^i \rangle C^i$ is player $i$'s end of period cash inflow generated by investing $C^i$ dollars at the beginning of the period in risky projects allocated across the projects according to the allocation vector, $\pi^i$.

We can think of each player's project portfolio formation problem as a club network formation problem in which each project, $q$, represents a club, and each connection from a player $i$ to a particular club, $q$, looks like the following:

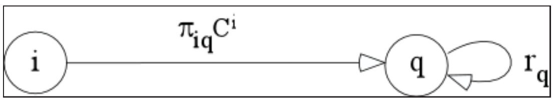


Figure 1: Player $i$ joins club $q \in Q$
and invests $\pi_{iq} C^i$ dollars in club-$q$

Using the language of networks, if player $i$ forms club network $\pi^i \in \Delta(Q)$ and operates his club network at intensity level $C^i$, then player $i$ allocates $\pi_{iq}$ fraction of his total intensity to club $q$. Thus, if realized project returns are $r := (r_1, \dots, r_m)$, player $i's$ payoff is $\langle r, \pi^i \rangle C^i$. This amount, together with loan repayments on loans made by player $i$ to

[2]For the moment we will simply assume players invest all of their cash flows in projects.

other players at the beginning of the period, are the only sources of player $i$'s end of period cash inflows. Because these sources of cash flow are risky, there is always some probability that these cash inflows will be insufficient to cover player $i$'s end of period contractual loan repayment obligations. These cash outflow obligations are the result of loans gotten by player $i$ (i.e., borrowing by $i$) from other players at the beginning of the period. In order to complete our network model, we must specify how these contractual obligations are resolved in the event that some player has insufficient funds. We will begin by modelling the process generating these risky cash flows. Our ultimate objective will be to model the origins of this risk generating process in the strategic behavior of the players in forming financial networks.

## 2.4 From Network Proposals to Networks

At the beginning of each period $[t, t+1]$, players observe the state, $\omega_t$, and therefore players know their cash flow levels, $C_t$, and based on this knowledge non-defaulted players, $i \in N \backslash D_t$, propose a feasible network,

$$((\pi_{iq})_{q \in Q}, (l^0_{ij})_{j \in N}, (l^1_{ij})_{j \in N}) \in \Delta(Q) \times \mathbb{B}(C^i_t) \times \mathbb{G}^i, \tag{23}$$

while each defaulted player, $i \in D_t$, has zero investable cash (i.e., $\max\{C^i_t, 0\} = 0$) and proposes the zero network, $(\pi^i_0, 0, 0)$, where portfolio $\pi^i_0 := (1, 0, \ldots, 0)$, requires that defaulted player $i$ invest all of his investable funds, $\max\{C^i_t, 0\}$, in project 0. In matrix form, players' proposed networks are given by a 3-tuple of matrices

$$G := (\Pi, L) := (\Pi, (L^0, L^1)) \in \mathbb{P}(C_t) := \Delta(Q)^n \times \mathbb{A}(C_t). \tag{24}$$

In order for the network proposed by the players to become the network in force during the coming period, the feasible loanable funds network proposal,

$$L := (L^0, L^1) \in \mathbb{A}(C) \tag{25}$$

must be feasible *and* matching, that is, $(L^0, L^1)$ must be contained in $\mathbb{AM}(C)$ (see assumptions [A-1]).

If the feasible proposed network, $G := (\Pi, L)$, is not matching, then players at the beginning of the period must bargain their way to a feasible *and matching* network whose representing matrix, $(\Pi', L')$, has rows such that

$$G' := (\pi'^i, l'^{0i}, l'^{1i})_{i \in N} \in \mathbb{G}(C_t) := \Delta(Q)^n \times \mathbb{AM}(C_t). \tag{26}$$

Here we will not model this bargaining process explicitly. Instead, we will assume that, given the $n$-tuple of cash flows, $C$, the outcome of the bargaining process is given by a of function,

$$\Gamma(C, \cdot) : \mathbb{P}(C) \longrightarrow \mathbb{G}(C) \text{ for } C \in M^n.$$

In Gong and Page (2016), for each $(C, G) \in M^n \times \mathbb{P}(C)$ the matching function, $(C, G) \longrightarrow \Gamma(C, G)$, is given by

$$\Gamma(C, G) := \begin{cases} G_0 := (\Pi_0, 0, 0) & \text{if } G \notin \mathbb{G}(C) \\ G := (\Pi, L^0, L^1) & \text{if } G \in \mathbb{G}(C). \end{cases} \tag{27}$$

Here, the $(0, 0)$ in the pair, $(\Pi_0, 0, 0) \in \Delta(Q)^n \times \mathbb{G}(C)$, is a pair of zero matrices - indicating that no borrowing or lending (and therefore no repayments) are being proposed or put in force. Also, $\Pi_0$ is the investments matrix where each row is given by $\pi^i_0 := (1, 0, \ldots, 0)$, indicating that each player invests all of his (positive) investable funds, $\max\{C^i, 0\}$, in project 0. Note that $(\Pi_0, 0, 0) \in \Delta(Q)^n \times \mathbb{G}(C)$ for all $C$ - i.e., $(\Pi_0, 0, 0)$ is feasible and matching at all investable funds levels - including the zero vector of investable funds. We will often write $(\Pi, L)$ rather than $(\Pi, L^0, L^1)$ - and sometimes will simply write $G$ rather than $(\Pi, L)$ and $G_0$ rather than $(\Pi_0, 0, 0)$, and we will often refer to $G_0$ as the *zero network*. Finally, we will call the matching function, $\Gamma(\cdot, \cdot)$, in expression (27) the $\Gamma$-*matching function*. We have the following formal definition:

**Definition 1** *(The $\Gamma$-Matching Function)*
*The $\Gamma$-matching function, $(C, G) \longrightarrow \Gamma(C, G)$, with domain $C \times \mathbb{P}(C)$, taking value in the space of feasible and matching networks, $\mathbb{G}(C)$, is given by*

$$(C, G) \longrightarrow \Gamma(C, G) := \begin{cases} G_0 := (\Pi_0, 0, 0) & \text{if } G \notin \mathbb{G}(C) \\ G := (\Pi, L^0, L^1) & \text{if } G \in \mathbb{G}(C). \end{cases}$$

The intuition behind the $\Gamma$-matching function is that if players fail to make matching proposals, then under the $\Gamma$-matching function players operate during the coming period under the consolation network, $G_0$. While $G_0$ is a feasible and matching network, there is in each state some feasible and matching network, $G \in \mathbb{G}(C)$, that Pareto dominates $G_0$ (see assumption [A-5] below). Thus, in equilibrium players are incentivized (via the $\Gamma$-matching function) to arrive at a feasible and matching network other than $G_0$. In this sense, the $\Gamma$-matching function acts as a penalty function that incentivizes players to propose feasible and matching networks.

The $\Gamma$-matching function can be rewritten compactly as follows: for all $G \in \mathbb{P}(C)$ proposed by players

$$\Gamma(C, G) := G I_{\mathbb{G}(C)}(G) + G_0(1 - I_{\mathbb{G}(C)}(G)) \in \mathbb{G}(C), \tag{28}$$

where $I_{\mathbb{G}(C)}(\cdot)$ is the indicator function for the set of affordable and matching networks.

**[A-3]** *(Matching Outcomes are Determined by the $\Gamma$-Matching Function)*
*In all cash flow states $C \in M^n$, we will assume that for all feasible network proposals, $G \in \mathbb{P}(C)$, the corresponding feasible and matching network is given by the value taken by the $\Gamma$-matching function,*

$$\Gamma(C, G) := \begin{cases} G_0 := (\Pi_0, 0, 0) & \text{if } G \notin \mathbb{G}(C) \\ G := (\Pi, L^0, L^1) & \text{if } G \in \mathbb{G}(C). \end{cases} \tag{29}$$

Note that for all $(C, G) \in M^n \times (\Delta(Q)^n \times \mathbb{G} \times \mathbb{G})$, if $G \notin \mathbb{G}(C)$, then under the $\Gamma$-matching function, all borrowing and lending stops and for the coming period players must invest all of their funds in the riskless project, that is,

$$\Gamma(C, G) = G_0 \in \mathbb{G}(C).$$

However, if $(C, G) \in Gr\mathbb{G}(C)$, then

$$\Gamma(C, G) = G \in \mathbb{G}(C).$$

Thus, because the zero network, $G_0$ is contained in $\mathbb{G}(C)$ for all $C \in M^n$, we have

$$\Gamma(Gr\mathbb{P}(C)) = \Gamma(Gr\mathbb{G}(C)). \tag{30}$$

Also note that in each cash flow state, $C \in M^n$, the set of feasible network proposals (matrices in this case), $\mathbb{P}(C)$, is a compact convex subset of $(R_+^{m+1})^n \times (R^n)^n \times (R^n)^n$, and more importantly, note that for all $C$ the set of all feasible and matching network proposals, $\mathbb{G}(C)$, is a closed convex subset of $\mathbb{P}(C)$. Thus, we have for all $C$,

$$\mathbb{G}(C) \subset \mathbb{P}(C) \subset (R_+^{m+1})^n \times (R^n)^n \times (R^n)^n.$$

Under the $\Gamma$-matching function, we have

$$\mathbb{G}(C) = \Gamma(C, \mathbb{P}(C)) \subset \mathbb{P}(C), \tag{31}$$

with

$$\Gamma(C, G) = G_0, \tag{32}$$

for all feasible but not matching network proposals, $G = \mathbb{P}(C) \backslash \mathbb{G}(C)$.

### 2.5 Networks, Cash Flows, and Contract Resolution

In this subsection, we will work out the end-of-period (i.e., short-term) cash flow consequences of the network chosen by the players given the cash available at the beginning of the period.

#### 2.5.1 Cash Flows Without Regard to Contract Resolution

Given players' cash flow, $C_{t-1}$, at time $t-1$, if the *prevailing feasible and matching network* during period $[t-1, t]$, is $G = (\Pi, L)$, and if the realized state at time point $t$ is $\omega_t$, then players' vector of cash flows, $C_t$, at the beginning of period $t+1$ (at time $t$) is given by

$$\begin{aligned} C_t := \quad & \underbrace{r_t \Pi^T I \overbrace{\left\{ \max\left[C_{t-1} - \overbrace{L^0 e}^{\text{net borrowing and lending at } t-1} \right], 0 \right\}}^{\text{investable funds}}}_{\text{return at } t \text{ in state } \omega_t \text{ from portfolio of projects invested in at } t-1} \\ & - \underbrace{L^1 e}_{\text{net repayments from borrowing and lending at } t-1}, \end{aligned} \tag{33}$$

where the levels of investable funds at time $t-1$ are given by the vector,

$$\left.\begin{aligned} &\max\left\{[C_{t-1} - L^0 e], 0\right\} \\ &:= \left([C_{t-1} - L^0 e] \vee 0\right) \\ &:= \left(\max\left\{[C_{t-1}^i - \langle l^{0i}, e\rangle], 0\right\}\right)_{i \in N} \\ &:= \left(\max\left\{[C_{t-1}^1 - \langle l^{01}, e\rangle], 0\right\}, \ldots, \max\left\{[C_{t-1}^n - \langle l^{0n}, e\rangle], 0\right\}\right) \end{aligned}\right\} \tag{34}$$

Thus, if the state at the beginning of period $t+1$ is $\omega_t$ (at time $t$) and if all players *meet their loan repayment obligations*, then players know their cash flows for the coming period and can compute their investable funds for the coming period corresponding to any borrowing-lending proposal they might make. If, with this knowledge, players form the network,

$$G(\omega_t) := (\Pi(\omega_t), L(\omega_t)) \in \mathbb{G}(C_t),$$

for the coming period, period $t+1$, then their network decisions, $G(\omega_t)$, in force during period $t+1$,together with their cash flows, $C_t$, the state of the real economy, $s_t$, the set of defaulted players, $D_t$, and the realized return on projects determine the cash flow, $C_{t+1}$, available at the beginning of period $t+2$ (at time $t+1$). In order for a player to participate in the network formation process, the player must have available at the beginning of the period a positive *(or at least nonnegative)* level of investable funds. If this is not the case for some player, say player $i^*$ then *that player must be able to borrow enough funds at the beginning of the period to meet his contractual repayment obligations.* Failing this, the player defaults and remains a defaulted player in perpetuity. Specifically, if at time $t$ the realized state $\omega_t$ is such that given the network formation decisions of the players at time $t-1$, player $i^{*\prime}s$ cash flow, $C_t^{i^*}$, going forward into period $t+1$, is nonpositive (i.e., $C_t^{i^*} < 0$), then player $i^*$, in order to continue playing the game of network formation and avoid becoming a defaulted player at time $t+1$, must be able to borrow sufficient funds to bring his investable funds level to a nonnegative amount - otherwise, the player defaults. Thus, if player $i^*$ at time $t$ in state $\omega_t$ has cash flow, $C_t^{i^*} < 0$, then player $i^*$ must propose a loanable funds network, $(l^{0i^*}(\omega_t), l^{1i^*}(\omega_t))$, such that

$$\underbrace{\left\langle r_t, \pi^{i^*}(\omega_{t-1})\right\rangle \left([C_{t-1}^{i^*} - \left\langle l^{0i^*}(\omega_{t-1}), e\right\rangle] \vee 0\right) - \left\langle l^{1i^*}(\omega_{t-1}), e\right\rangle}_{C_t^{i^*}} - \left\langle l^{0i^*}(\omega_t), e\right\rangle \geq 0.$$

If this is not possible - if player $i^*$ fails to borrow his way back to solvency because no other players will take the lending side of his borrowing proposals, $l^{i^*}(\omega_t) := (l^{0i^*}(\omega_t), l^{1i^*}(\omega_t))$, (i.e., if player $i^*$ is unable to find lenders willing to supply the needed funds), or if player $i^*$ is unwilling to make such a borrowing proposal, then player $i^*$ will become a permanent member of the defaulted players club (starting at time $t+1$ - i.e., the player has one period to make good on his debt obligations). By the constraint mapping, $C^i \longrightarrow \mathbb{B}(C^i)$, a player whose cash flow is negative is constrained to only make borrowing proposals or no proposals at all.

### 2.5.2 Contract Resolution

The problem above is even a bit more subtle than what we have described. In particular, if a player has insufficient funds to meet his contractual repayment obligations, this may have the unintended consequence of causing other players to have insufficient funds to meet their repayment obligations. In this subsection we will work out, using the results of Eisenberg and Noe (2001), precisely *what players are able to repay whenever some player cannot meet his contractual repayment obligations.* In fact, the Eisenberg and Noe approach will provide us with a stationary allocation rule for the resolution of contractual default issues - and thus, will allow us to calculate an accurate measure of the cash flow externalities caused by a player's insolvency as well as give us a accurate read concerning how much a player must borrow in order to get back to solvency.

We begin by rewriting the fundamental cash flow transition equation (33) by re-representing the contractual repayments matrix, $L^1$, using a liabilities allocation matrix (as in Eisenberg and Noe, 2001) together with the contractual repayments vector. Thus decomposing the repayment matrix $L^1$ into inflows and outflows. Let

$$\underbrace{m_{ij} = \frac{\max\{l_{ij}^1, 0\}}{\sum_j \max\{l_{ij}^1, 0\}}}_{\text{fraction player } i's \text{ liabilities owed to player } j \text{ (from i to j)}} \tag{35}$$

Consider the *liabilities allocation matrix*,

$$M := \begin{pmatrix} m_{11} & \cdots & m_{1j} & \cdots & m_{1n} \\ \vdots & & \vdots & & \vdots \\ m_{i1} & \cdots & m_{ij} & \cdots & m_{in} \\ \vdots & & \vdots & & \vdots \\ m_{n1} & \cdots & m_{nj} & \cdots & m_{nn} \end{pmatrix} \tag{36}$$

with $j^{th}$ column $m_j$ and $i^{th}$ row $m^i$. Note that for any state $\omega \in \Omega$, the liabilities allocation matrix, $M$, is matching, i.e., $M \in \mathbb{M}$. Letting

$$O^i := \sum_j \max\{l^1_{ij}, 0\}, \tag{37}$$

$O^i$ is the total contractual loan repayment amount by player $i$ to all other players. Finally, let,

$$O := (O^1, \ldots, O^n), \tag{38}$$

be the vector of total contractually specified loan repayment amounts by each of the players.

If all players have sufficient funds to cover their contractual loan repayment obligations, then player $j$ will receive an amount given by

$$\langle O, m_j \rangle \tag{39}$$

from all other players to which player $j$ loaned money at the beginning of the period at time $t-1$. Thus, if all players are solvent, the vector of repayments from players who were loaned funds at $t-1$ to players who did the lending, is given by

$$M^T O = \begin{pmatrix} \langle O, m_1 \rangle \\ \vdots \\ \langle O, m_j \rangle \\ \vdots \\ \langle O, m_n \rangle \end{pmatrix}. \tag{40}$$

Under network $(\Pi, L)$, if all players remain solvent, players' cash flow vector $C_t$ in state $\omega_t$ is given by

$$\begin{aligned} C_t &:= r_t \Pi^T I \left( [C_{t-1} - L^0 e] \vee 0 \right) - L^1 e \\ &:= r_t \Pi^T I [C_{t-1} - L^0 e] + \underbrace{M^T O - O}_{-L^1 e}. \end{aligned}$$

Thus, the fundamental cash flow transition equation (with no insolvency) becomes

$$\left. \begin{array}{rcl} \underbrace{C_t}_{\text{players' cash flows at } t} & := & \underbrace{r_t \Pi^T I \left( [C_{t-1} - L^0 e] \vee 0 \right)}_{\text{cash inflow to } i \text{ from returns on portfolio}} \\ + \underbrace{M^T O}_{\text{loan repayments received}} & - & \underbrace{O}_{\text{loan repayments paid out}} . \end{array} \right\} \tag{41}$$

### 2.5.3 Payment Vectors and Clearing Equilibrium

Under the assumptions of absolute priority and limited liability in contract resolution, we can apply Eisenberg and Noe (2001) to calculate a vector whose components are the amounts each player is able to pay (prior to additional borrowing) - thus giving us the true amount of the shortfall that must be made up by an insolvent player in order to avoid default. We call this new vector of repayments, $p$, the clearing vector. Because our dynamic model of network formation is stationary (i.e., the primitives are time independent), using results in Eisenberg and Noe (2001), we will be able to write our clearing vector, $p$, as a stationary (i.e., time invariant) continuous function of the underlying parameters and player choices: project returns, $r$, contractually specified repayment obligations, $O$ (a continuous function of $L^1$), cash flows, $C$, investment club networks $\Pi$, and loanable funds networks, $L^0$ and $L^1$ (i.e., $M^1$ and $O$). Thus, ex ante, realizable repayments are given by a parameterized set of functions given by,

$$\{p(\cdot, C, G) : G \in \mathbb{G}(C)\},$$

where for each network $G = (\Pi, L^0, L^1) \in \mathbb{G}(C)$, the vector-valued function, $r \longrightarrow p(r, C, G)$, is a jointly continuous function of $(r, C, G)$. In particular, following Eisenberg and Noe (2001), we say that a payment vector $p$ - given project returns (at $t+1$) $r$, cash flow levels (or cash at $t$), $C$, and network $G = (\Pi, L^0, L^1)$ (formed at $t$ and in force during period $[t, t+1]$ - is a clearing equilibrium provided

$$\begin{aligned} p &= \min\{r\Pi^T I\left([C - L^0 e] \vee 0\right) + pM^T, O\} \\ &:= \left(r\Pi^T I\left([C - L^0 e] \vee 0\right) + pM^T\right) \wedge O \\ &:= \Phi(r\Pi^T I\left([C - L^0 e] \vee 0\right) + pM^T, O).^3 \end{aligned}$$

Thus, $p$ is a fixed point of the vector function,

$$p \longrightarrow \Phi(r\Pi^T I\left([C - L^0 e] \vee 0\right) + pM^T, O) : [0, M_+^n] \longrightarrow [0, M_+^n].$$

Given $(r, C, G) := (r, C, (\Pi, L^0, L^1))$, $p$ is a clearing equilibrium if and only if

$$p = r\Pi^T I\left([C - L^0 e] \vee 0\right) + pM^T \wedge O.$$

Moreover, given cash flow-network pair, $(C, G)$, $p(r, C, G)$ is a clearing equilibrium for any return vector $r$ realized at the end of any period in which the cash flow vector at the beginning of the period is $C$ and the network in force during the period is $G$. Thus, we have

$$p(r, C, G) = \Phi(r\Pi^T I\left([C - L^0 e] \vee 0\right) + p(r, C, G)M^T, O).$$

It follows from Eisenberg and Noe (2001) that for each $(r, (C, G)) \in \mathcal{R} \times Gr\mathbb{G}(\cdot)$ a clearing equilibrium *exists* and is *unique* provided the

project returns are strictly positive with probability 1.

Thus, while project returns may be arbitrarily low, they must be positive with probability 1.

Let $p(r, (C, G))$ denote the unique clearing equilibrium payment vector given $(r, (C, G))$. It follows from Eisenberg and Noe (2001) that the clearing equilibrium payment function, $p(\cdot, (\cdot, \cdot))$, is jointly continuous in $(r, (C, G))$, and that each of the functions,

$$r \longrightarrow p(r, (C, G)) \text{ and } C \longrightarrow p(r, (C, G)),$$

[3] Both the matrix. $M^T$, and the vector, $O_t$, are functions of the repayment matrix, $L^1$ (see expressions, (35) and (36)).

is concave, increasing and nonexpansive, separately. Thus, $p(\cdot,(C,G))$ is concave, increasing, and nonexpansive in project returns, $r$, for each $(C,G)$, and $p(r,(\cdot,G))$ is concave, increasing, and nonexpansive in cash flow levels, $C$, for each $(r,G)$.

The new adjusted fundamental cash flow transition equation (allowing insolvency) is

$$\left.\begin{array}{cccc} & \underbrace{C_{t+1}}_{\text{players' cash flows at } t+1} & := & \underbrace{r_{t+1}\Pi^T I\left([C_t - L^0 e]\vee 0\right)}_{\text{cash inflow to } i \text{ from returns on portfolio}} \\ + & \underbrace{p(r_{t+1},C_t,G)M^T}_{\text{adjusted loan repayments received}} & - & \underbrace{p(r_{t+1},C_t,G)}_{\text{adjusted loan repayments paid out}} \quad . \end{array}\right\} \tag{42}$$

### 2.5.4 Player's Cash Flows, Contract Resolution, and Player Default

From the new adjusted fundamental cash flow transition equation (42), we obtain for each player $i$ a new adjusted fundamental equation for cash flows given by

$$\left.\begin{array}{cccc} & \underbrace{C^i_{t+1}}_{i's \text{ cash flow at } t+1} & := & \underbrace{\left\langle r_{t+1},\pi^i\right\rangle\left([C^i_t - \left\langle l^{0i},e\right\rangle]\vee 0\right)}_{\text{cash inflow to } i \text{ from returns on portfolio}} \\ + & \underbrace{\left\langle p(r_{t+1},C_t,G),m_i\right\rangle}_{\text{adjusted loan repayments received}} & - & \underbrace{p^i(r_{t+1},C_t,G)}_{\text{adjusted loan repayments paid out}} \quad . \end{array}\right\} \tag{43}$$

In order for the network formation process to work, the investable funds, $C^i_t - \left\langle l^{0i},e\right\rangle$, available to each player, $i$, at the beginning of each period, $[t,t+1]$, must be nonnegative. If this is not the case for some player, that player must be able to borrow enough funds at the beginning of the period to make up the short fall in investable funds. Specifically, if at time $t$ the realized state $\omega_t$ is such that given the network formation decisions of the players at time $t-1$, some player's level of cash flow, going forward into period $t+1$, is negative, then this player, say player $i^*$, in order to continue playing the game of network formation and avoid becoming a defaulted player, must be able to borrow sufficient funds to bring his investable funds level to a nonnegative amount - otherwise the player defaults. Thus, if player $i^*$ at time $t+1$ in state $\omega_{t+1}$ has investable funds, $C^{i^*}_{t+1} < 0$, then player $i^*$ must propose a loanable funds network, $(l^{0i^*}(\omega_{t+1}), l^{1i^*}(\omega_{t+1}))_{j\in N}$, such that

$$\left[C^{i^*}_{t+1} - \left\langle l^{0i^*}(\omega_{t+1}),e\right\rangle\right] \geq 0, \tag{44}$$

where $C^{i^*}_{t+1}$ is given by expression (43). If player $i^{*\prime}s$ efforts to borrow his way back to solvency fail - if player $i^*$ is unable to find lenders willing to supply the needed funds (i.e., the amount $-\left\langle l^{0i^*}(\omega_{t+1}),e\right\rangle$ in expression 44), then player $i^*$ will become a permanent member of the defaulted players club.

## 2.6 The Cash Flow Transition Function

At any time $t$ cash flows at $t+1, C_{t+1}$, are a function of current cash flows, $C_t$, players's current actions, $G_t := (\Pi_t, L^0_t, L^1_t)$, the stationary default adjustment function, $p(\cdot,\cdot,\cdot)$, and risky project returns at $t+1$, $r_{t+1}$. Thus, rewriting expression (43) as an abstract vector function we have

$$C_{t+1} = F_p(C_t, G_t, r_{t+1}). \tag{45}$$

If the current state is $\omega_t := (C_t, D_t, s_t)$, then player $i's$ immediate expected payoff is

$$r_i(\omega_t, G_t) := \int_{\mathcal{R}} u_i(F_p(C_t, G_t, r_{t+1}))g(dr_{t+1}|D_t, s_t), \tag{46}$$

where

$$\left\{g(dr_{t+1}|D_t, s_t) : (D_t, s_t) \in 2^N \times S\right\} \tag{47}$$

is the collection of product measurable probability density functions describing the random behavior of project returns as a function of the state of the real economy and the set of defaulted players.

## 2.7 The Short-Term Lending Rate Process

Consider the network formation process,

$$\{G(\omega_t)\}_t := \{\Pi(\omega_t), L^0(\omega_t), L^1(\omega_t)\}_t, \tag{48}$$

with underlying equilibrium state process,

$$\{\omega_t\}_t := \{(C_t, D_t, s_t)\}_t, \tag{49}$$

governed by the equilibrium Markov transition kernel, $p^*(d\omega_{t+1}|\omega_t)$. For the equilibrium borrowing-lending-repayment network formation process, $\{L(\omega_t)\}_t := \{L^0(\omega_t), L^1(\omega_t)\}_t$, the short-term nominal lending rate process, $\{[int_{ij}(\omega_t)]_{ij}\}_t$, is given by,

$$int_{ij}(\omega_t) := \frac{l^1_{ji}(\omega_t)}{l^0_{ij}(\omega_t)} - 1, \tag{50}$$

where for all $t$ and $\omega_t$, $l^0_{ij}(\omega_t)$ is the amount of the loan from player $i$ to player $j$, and $l^1_{ji}(\omega_t)$ is the nominal loan repayment amount from player $i$ to player $j$.

The equilibrium borrowing-lending-repayment network formation process,

$$\{L^0(\omega_t), L^1(\omega_t)\}_t,$$

induces an equilibrium default adjustment process given by,

$$\{p(r_{t+1}, C(\omega_t), G(\omega_t))\}_t := \{p^1(r_{t+1}, C(\omega_t), G(\omega_t)), \ldots, p^n(r_{t+1}, C(\omega_t), G(\omega_t))\}_t. \tag{51}$$

Given the equilibrium processes, $\{\omega_t, G(\omega_t)\}_t$ and $\{p(r_{t+1}, C(\omega_t), G(\omega_t))\}_t$, the induced equilibrium short-term, default-adjusted lending rate process, $\{[p\text{-}int_{ij}(\omega_t)]_{ij}\}_t$, is given by,

$$p\text{-}int_{ij}(\omega_t) := \frac{m_{ji}(\omega_t)[\int_{\mathcal{R}} p^j(r_{t+1}, C(\omega_t), G(\omega_t))g(dr_{t+1}|D_t, s_t)]}{l^0_{ij}(\omega_t)} - 1, \tag{52}$$

where for all $t$ and $\omega_t$, $m_{ji}(\omega_t)$ is the fraction of player $j's$ short-term liabilities owed to player $i$, and $\int_{\mathcal{R}} p^j(r_{t+1}, C(\omega_t), G(\omega_t))g(dr_{t+1}|D_t, s_t)$ is the expected default adjustment at $t$ in state $\omega_t := (C_t, D_t, s_t)$. In the case of no defaulted players (i.e., $D_t \cup N_f(\omega_t) = \varnothing$), the equilibrium short-term, default-adjusted lending rate process, $\{[p\text{-}int_{ij}(\omega_t)]_{ij}\}_t$, is equal to the short-term nominal lending rate process, $\{[int_{ij}(\omega_t)]_{ij}\}_t$. Thus, we have,

$$p\text{-}int_{ij}(\omega_t) := \frac{l^1_{ji}(\omega_t)}{l^0_{ij}(\omega_t)} - 1. \tag{53}$$

In general, the equilibrium default risk premium is given by

$$p\text{-}premium_{ij}(\omega_t) := \frac{l^1_{ji}(\omega_t) - m_{ji}(\omega_t)[\int_{\mathcal{R}} p^j(r_{t+1}, C(\omega_t), G(\omega_t))g(dr_{t+1}|D_t, s_t)]}{l^0_{ij}(\omega_t)}. \tag{54}$$

# 3 A Definition of Systemic Risk

For the convenience of the reader, we begin with a brief summary of some of the classical notions from the theory of Markov chains needed for the discussion of the stability properties of network dynamics (see also, Gong, Page, and Wooders, 2015). These classical notions will help us greatly in coming to deeper understanding of the dynamic behavior of systemic risk and its origins in the strategic behavior of the players in the game of temporary financial network formation. In the next section we present our discounted stochastic game model of temporary financial network formation - thereby offering our view of the strategic foundations of systemic risk.

## 3.1 Some Classical Notions from the Theory of Markov Chains

Let,

$$\{\omega_t\}_t := \{(C_t, D_t, s_t)\}_t,$$

be the equilibrium state process governed by the equilibrium Markov transition kernel, $p^*(d\omega_{t+1}|\omega_t)$. The $t$-step transition $p^{*t}(\cdot|\cdot)$ is defined recursively as follows: for all $\omega := (C, D, s) \in \Omega$ and $E \in B_\Omega$,

$$p^{*t}(E|\omega) = \int_\Omega p^*(E|\omega')p^{*t-1}(d\omega'|\omega) = \int_\Omega p^{*t-1}(E|\omega')p^*(d\omega'|\omega), \tag{55}$$

for $t = 1, 2, \ldots$, and $p^{*0}(\cdot|\omega) = \delta_\omega(\cdot)$ is the Dirac measure at $\omega$.

Also, for now we will assume that the equilibrium Markov transition kernel, $p^*(\cdot|\cdot)$, is *uniformly countably additive* on the entire state space (we will formally establish this fact below). This implies, via results due to Tweedie (2001), that the state space is decomposable into finitely many *basins of attraction* (i.e., largest absorbing sets),

$$\mathcal{A} := \{H^1, H^2, \ldots, H^L\},$$

and a transient set $T$. Thus, because $p^*(\cdot|\cdot)$ is globally uniformly countably additive on all of $\Omega$, we can write,

$$\Omega = \left[\cup_{l=1}^L H^l\right] \cup T.$$

This decomposition of the state space is usually called (in Markov chain theory) a *Harris decomposition*.

For the state process $\{\omega_t\}_{t=1}^\infty = \{(C_t, D_t, s_t)\}_{t=1}^\infty$, the *hitting time* (or first passage time) to

$$H := \left[\cup_{l=1}^L H^l\right] \in B_\Omega$$

(or for that matter the hitting time to any set $H \in B_\Omega$) is given by

$$\tau_H := \inf\{t \geq 1 : \omega_t \in H\}. \tag{56}$$

Following Tweedie (2001) and Meyn and Tweedie (2009), the probability (i.e., the *first passage probability*) that, starting from state $\omega$, the state process *"hits" H in finite time* is given by

$$L(\omega, H) := \mu\{\tau_H < \infty|\omega_0 = \omega\} = \mu\{\cup_{t=1}^\infty (\omega_t \in H|\omega_0 = \omega)\}. \tag{57}$$

Moreover, the probability that, starting from state $\omega$, the state process "hits" $H$ at exactly time $t$ is given by

$$\mu\{\tau_H = t|\omega_0 = \omega\} = \int_{H^c} p^*(d\omega_1|\omega_0)\int_{H^c} p^*(d\omega_2|\omega_1)\cdots\int_{H^c} p^*(H|\omega_{t-1})p^*(d\omega_{t-1}|\omega_{t-2}). \tag{58}$$

Finally, the probability that, starting from state $\omega$, the state process "hits" $H$ at any one time point $t = 1, 2, 3, \ldots, t'$ between now and $t'$ is given by

$$\mu \{\tau_H \leq t' | \omega_0 = \omega\} = \sum_{t=1}^{t'} \mu \{\tau_H = t | \omega_0 = \omega\} . \tag{59}$$

Because the Markov transition kernel, $p^*(\cdot|\cdot)$, is globally uniformly countably additive we also know from Tweedie (2001) that not only does the state space have a Harris decomposition, $\left[\cup_{l=1}^{L} H^l\right] \cup T$ but also that

$$L(\omega, \left[\cup_{l=1}^{L} H^l\right]) = 1 \text{ for all } \omega \in \Omega.$$

Thus, no matter where the state process starts, in finite time with probability 1 the process will reach one of finitely many basins of attraction - and once there will stay there.

Finally, letting

$$P_D(N) := \{D' \in 2^N : D \subseteq D'\} \tag{60}$$

(i.e., the collection of all subsets of players containing the subset $D$), we will assume that the equilibrium Markov transition kernel is such that for all $\omega := (C, D, s) \in \Omega$,

$$p^*(M^n \times P_D(N) \times S|\omega) = 1. \tag{61}$$

Thus, recalling expression (10) specifying the set of players who qualify for permanent membership in the set of defaulted players (starting next period) in current state $\omega$,

$$N_f(\omega) := \left\{i \in N : C^i(\omega) \leq 0 \text{ and } C^i(\omega) - \left\langle l^{0i}(\omega), e\right\rangle \leq 0\right\},$$

(61) implies that

$$N_f(\omega) \subseteq N_f(\omega') \subseteq N_f(\omega''), \tag{62}$$

for the coming state $\omega'$ and for all future states $\omega''$. Because each basin of attraction is Harris recurrent and irreducible, expression (62) implies that if $\omega = (C, D, s) \in H^l$, then for any other state $\omega' = (C', D', s') \in H^l$, the set of defaulted players $D$ and $D'$ are equal. Thus, basin are homogeneous with respect to their default characteristics. We will return to this fact about basins in the next section when we present our stochastic game model of financial network formation.

## 3.2 A Definition of Systemic Risk Based on the Dynamics of Temporary Financial Networks

Recall that for a non-defaulted player $i$ the set of states in which player $i$ qualifies for permanent membership in the set of defaulted players is given by,

$$\Omega_f^*(i) := \left\{\omega \in \Omega : C^i(\omega) \leq 0 \text{ and } C^i(\omega) - \left\langle l^{0i*}(\omega), e\right\rangle \leq 0\right\}, \tag{63}$$

where $C^i(\omega) := proj_{M^i}(\omega) = C^i \in M^i$. Regarding the non-defaulted player, an important question to ask is, given the equilibrium behavior of the players - as captured by the network-valued equilibrium stochastic process, $\{G^*(\omega_t)\}_t$, what is the probability that at some time point in the future an as yet non-defaulted player, $i \in N \backslash D_t$, will join the set of defaulted players? This is equivalent to asking, given the underlying equilibrium state process, $\{\omega_t\}_t$, what is the probability that the state process reaches in finite time a state contained in $\Omega_f^*(i)$. We note that this is a question about the first passage probability - or the hitting probability - for the set of states in which player $i$ will qualify for default (or join the set of defaulted players). With this basic observation in mind -

and assuming the equilibrium state process, $\{\omega_t\}_t := \{(C_t, D_t, s_t)\}_t$ , is governed by a globally countably additive Markov transition kernel, $p^*(\cdot|\cdot)$, with basins of attraction, $\mathcal{A} := \{H^1, H^2, \ldots, H^L\}$, what can we say - and more importantly, what can we learn about systemic risk?

While there is no generally agreed upon formal definition of systemic risk (a fact noted and discussed in Glasserman and Young 2015, section 6), there seems to be a widespread intuitive understanding of it, and researchers can easily point to real world examples of it (e.g., the financial crisis of 2008). An informal description of systemic risk would go something like the following: *Systemic risk is a measure of the conditional likelihood that a system in a particular state will fail if a particular event occurs.* Usually, by a particular event, we mean a "shock". Thus if the system is in a particular state and if there is a shock to the system, then systemic risk is the conditional likelihood that the system will fail. At the very outset there are three terms in our word definition which must be given precise meanings if we are to provide a formal definition of systemic risk. The first term is "system" - what is a system? Here, we represent the "system" as a "network" - a notion to which we have given a precise definition. The second term is "state" - what does it mean for the system to be in a particular state. Here again we have given a precise definition of what we mean by a state, as well as what we mean by a network being in a particular state. The third term is "fail" - what do we mean by fail? Here, we can substitute the term default for the term fail. Since a financial network is made of many interconnected nodes (representing players), what does it mean for the network to fail? What constitutes a failure? Does it mean that all players (nodes) default, does it mean that some players default, or does it mean that *sufficiently many* players default to cause the financial network to cease functioning. So there are degrees or gradations of network failure. Here we can take this into account in a very precise way by keeping track of the set of players who default. There is also the question of what player in the network receives the shock - and how does a player receive the shock? Where does the shock come from? - is it generated by the network? Here, shocks are endogenously generated via project returns (where projects are nodes in our network). Who receives (i.e., which players receive) the shock is therefore determined entirely by the network structure endogenously chosen by the players (the nodes). And what is a shock? Here a shock can be viewed as an extreme realization of project returns. And finally, there are "things" missing from the intuitive definition. Most notably, time. Are we thinking about an immediate failure of the network due to some shock? - or, are we thinking about the eventual failure of the network? Our definition of systemic risk will take into account the timing as well as the severity of the failure.

In addition to these considerations, our definition of systemic risk is unique in that it is based upon the strategic behavior of the players in the game of network formation and is given explicitly as a function of the underlying equilibrium dynamics. These equilibrium dynamics are, in turn, determined by the stationary Markov equilibrium in players' network formation strategies, $\omega_t \longrightarrow G^*(\omega_t)$, which emerge from the interplay between the strategic best response behavior of the players, the changing network structure, and risk. In our view, any measure of systemic risk which fails to take into account equilibrium dynamics and the feedback between strategic behavior, network structure, and risk will fail to accurately measure the risk of systemic failure.

We can think of the paths that the equilibrium stochastic state process, $\{\omega_t\}_t$, can follow in reaching a particular subset of states, say for example $\Omega^*_f(i)$ (see expression 63) above, as part of a larger "transportation" network over which the equilibrium state will travel in moving from one particular state to a set of different states. *Carrying the transportation analogy of equilibrium state dynamics further, we are lead to define endogenous systemic risk as the probability that the equilibrium stochastic state process,* *starting at a given state arrives at a subset of failed states (e.g., $\Omega_f^*(i)$), at or before a given time.* Thus, we define endogenous systemic risk as the *equilibrium first passage probability* to a subset consisting of failed states starting from a given state - where what we mean by failed states is specified by the requirement that the set of defaulted players be of a certain composition and size. Our notion of endogenous systemic risk, therefore, is one inextricably linked to the equilibrium state process determined by the underlying discounted stochastic game of network formation. Following our approach, rather than there being a single measure of systemic risk, there is instead a *schedule of systemic risk measures* which lists the probabilities of various arrival times at various subsets of failed states, departing from any given state.[4] Now, for the details.

Given the equilibrium state process, $\{\omega_t\}_t := \{(C_t, D_t, s_t)\}_t$, for each possible nonempty subset of players, $S \in 2^N \backslash \varnothing$, of players, let

$$\Omega_f^*(S) := \left\{\omega \in \Omega : C^{i*}(\omega) \leq 0 \text{ and } C^{i*}(\omega) - \left\langle l^{0i*}(\omega), e\right\rangle \leq 0 \text{ for } i \in S \;\right\} := \cap_{i \in S} \Omega_f^*(i).$$

$\Omega_f^*(S)$ is the subset of states in which all players in $S$ join the set of permanently defaulted players, starting next period. Thus, if $\omega_t \in \Omega_f^*(S)$, where $\omega_t = (C_t, D_t, s_t)$, then we know that

$$S \subseteq N_f^*(\omega_t) \text{ and } D_{t+1} = D_t \cup N_f^*(\omega_t)$$

where recall $N_f^*(\omega_t)$ is the set of all players who qualify for permanent membership in the set of defaulted players in state $\omega_t$. We also know that $D_t \cup S \subset D_{t+1}$ and we know that the new entrants into the set of defaulted players during period $[t, t+1]$ are $N_f^*(\omega_t) \backslash D_t$. Thus, $D_{t+1} = D_t \cup N_f^*(\omega_t)$.

Our definition of endogenous systemic risk is the following:

**Definition 2** *(Endogenous Systemic Risk)*
*Given the equilibrium state process, $\{\omega_t\}_t := \{(C_t, D_t, s_t)\}_t$, the systemic risk that a subset of non-defaulted players, $S$, in financial network $G^*(\omega_0) = G^*(C_0, D_0, s_0)$, $S \subset N \backslash D_0$, all qualify for default at exactly time $t'$, denoted by $SR_{t'}(\omega_0, \Omega_f^*(S))$, is given by*

$$SR_{t'}(\omega_0, \Omega_f^*(S)) = \mu\left\{\tau_{\Omega_f^*(S)} = t | \omega_0 = \omega\right\}.$$

*The systemic risk that players $S$ all qualify for default at any time between now and $t'$, denoted by $SR_{[1,t']}(\omega_0, \Omega_f^*(S))$, is given by*

$$SR_{[1,t']}(\omega_0, \Omega_f^*(S)) = \sum_{t=1}^{t'} \mu\left\{\tau_{\Omega_f^*(S)} = t | \omega_0 = \omega\right\}.$$

*The systemic risk that players $S$ all qualify for default ever, denoted by $SR_{[1,\infty)}(\omega_0, \Omega_f^*(S))$, is given by*

$$SR_{[1,\infty)}(\omega_0, \Omega_f^*(S)) = \mu\left\{\tau_{\Omega_f^*(S)} \leq \infty | \omega_0 = \omega\right\} = \sum_{t=1}^{\infty} \mu\left\{\tau_{\Omega_f^*(S)} = t | \omega_0 = \omega\right\}.$$

## 3.3 The Dynamics of Systemic Risk: Basins and Their Spheres of Influence

The presence of basins of attraction has profound implications for our understanding of the dynamic behavior of systemic risk, and therefore, for our ability to measure and

[4] For an excellent survey of what is known about systemic risk with respect to other definitions of systemic risk, see Glasserman and Young (2015) - and for an excellent discussion of other notions of endogenous systemic risk see Zigrand (2014).

control systemic risk. This is true for two reasons. First, as noted above, the basins of attraction generated by the equilibrium state process are homogenous with respect to their default characteristics - meaning, that if a basin contains a state, $\omega := (C, D, s)$, with defaulted player set, $D$, then all states in the basin will have precisely the same set of defaulted players - no more, no less. The existence of basins and the stratification of basins by default characteristics reduces the difficulty of making a global assessment of systemic risk in any financial network. Given the current position of the network in the "transportation system" and the allocation of defaulted player sets across basins, the systemic risk of (or the first passage probability to) each of the finitely many basins containing defaulted players can easily be calculated. The big picture take away from our approach is that *what is critical in assessing systemic risk in any risky financial network is the allocation of default levels across the basins of attraction*.

Second, given the equilibrium state dynamics, the profile of basins of attraction these dynamics generate comes equipped with a unique set of *tipping points* and *spheres of influence*. Here, using our definition of systemic risk, we will formally define the notions of tipping point and sphere of influence. Intuitively, once the state processes reaches a tipping point, then with probability 1, the coming state will be contained in the sphere of influence of one of two possible basins (possibly a good basin versus a bad basin). Thus, each tipping point state is the gateway to two different spheres of influence belonging to two different basins. Once the process enters a sphere of influence belonging to a particular basin, the process has passed the point of no return with regard to its eventual arrival in that particular basin, and thus the sequence of future states will lead inexorably to that particular basin. If the default characteristics of that basin are extremely bad, then this sequence of future states might best be described as a default cascade. Tipping points, therefore, are the state dynamic's early warning system of impending entry into a sphere of influence - and if that sphere belongs to a bad basin - of impending doom. A tipping point that is a point of departure for the state's journey to a severely failed basin is truly a *systemic state* - and such systemic states can be easily identified.

Because the set of defaulted players persists and is nondecreasing and because the equilibrium state process, once it enters a basin of attraction stays in the basin visiting each state in the basin infinitely often, it follows that if the basin, say basin $H^l$, contains a state, $\omega := (C, D, s)$, with defaulted player set, $D$, such that $|D| = k_l$, then all states in the basin will have precisely the same set of defaulted players consisting of $k_l$ many defaulted players. We say that such a basin, $H^l$ is a $k_l$-level basin. We summarize these observations in the following Theorem.

**Theorem 1** *(Homogeneity of Default Levels of Basins of Attraction)*
*Let $\{\omega_t\}_t$ be the state process governed by Markov transition kernel, $p^*(\cdot|\cdot)$, having basins of attraction*

$$\mathcal{A} := \left\{H^1, H^2, \ldots, H^L\right\}.$$

*For each basin, $H^l$, there is a unique subset of defaulted players, $D^l$, such that for all $\omega \in H^l$, $proj_{2^N}(\omega) = D^l$. Moreover, in state $\omega_t$ at time $t$, if $p^*(H^l|\omega_t) = 1$, then $D^l = D_t \cup N_f^*(\omega_t)$.*

Let $\Omega_f \subset \Omega$ denote the collection of states in which there are defaulted players (i.e., states in which the set of defaulted players is nonempty). We have

$$\Omega_f := \{\omega \in \Omega : D(\omega) \neq \varnothing\} = \{\omega \in \Omega : |D(\omega)| > 0\}, \tag{64}$$

where $D(\omega) := proj_{2^N}(\omega) = D \in 2^N$. We will refer to the states contained in $\Omega_f$ as default states. Refining this a bit, let

$$\Omega_f^k := \{\omega \in \Omega : |D(\omega)| = k\}, \tag{65}$$

denote the set of states in which there are exactly $k = 0, 1, \ldots, n$ defaulted players. We will refer to the states contained in $\Omega_f^k$ as k-states. Next, let $B := \{1, 2, \ldots, L\}$, be index set for the set of basins of attraction, $\mathcal{A}$, and let

$$B_f := \{l \in B : \Omega_f \cap H^l \neq \varnothing\},$$
and
$$B_c := \{l \in B : \Omega_f \cap H^l = \varnothing\}.$$

Consider subsets of states given by

$$H_f := \cup_{l \in B_f} H^l \text{ and } H_c := \cup_{l \in B_c} H^l. \tag{66}$$

Note that if $l \in B_f$, then $\Omega_f^k \cap H^l \neq \varnothing$ for some $k \in \{0, 1, \ldots, n\}$, while, due to the homogeneity of the default characteristics of basins, $\Omega_f^{k'} \cap H^l = \varnothing$ for $k' \neq k$. We will call such a basin a $k$-level basin. Alternatively, if $l \in B_c$, then $H^l$ is free of defaulted players. Thus, the state space can be partitioned as follows:

$$\Omega = T \cup H_c \cup H_f,$$

where for each transient state $\omega \in T$,

$$SR^*_{[1,\infty)}(\omega, H_c) + SR^*_{[1,\infty)}(\omega, H_f) = 1$$

If at $\omega \in T$,

$$SR^*_{[1,\infty)}(\omega, H_c) = 1,$$

then default is no longer possible because in finite time with probability 1 the state process will enter a basin of attraction containing no defaulted players - and will remain there for all future periods. However, if at $\omega \in T$,

$$SR^*_{[1,\infty)}(\omega, H_f) = 1,$$

then default is inevitable because in finite time with probability 1 the state process will enter a basin of attraction containing states having some fixed subset of defaulted players.

**Definition 3** *(A Basin's Sphere of Influence)*
*Let $\{\omega_t\}_t$ be the state process governed by Markov transition kernel, $p^*(\cdot|\cdot)$, having basins of attraction*

$$\mathcal{A} := \left\{H^1, H^2, \ldots, H^L\right\}.$$

*For each basin, $H^l$, there is a sphere of influence given by*

$$SI^*(H^l) := \left\{\omega \in \Omega : SR^*_{[1,\infty)}(\omega, H^l) = 1\right\}. \tag{67}$$

If the process reaches state, $\omega$, contained in the sphere of influence, $SI^*(H^l)$, then in finite time with probability 1 the process will enter basin $H^l$. If $H^l$ is a $k$-level basin, $k > 0$, then all states in $H^l$ will have a set of defaulted players of size $k$. Thus, a $k$-level default is inevitable once state $\omega \in SI^*(H^l)$ is reached. If $\omega \in SI^*(H^l)$ where $H^l$ is a $k$-level basin, $k > 0$, then all state paths starting with $\omega$ are referred to as $k$-level default cascades.

The sphere of influence of any basin can be small. In fact, it is even possible that a basin is its own sphere of influence. Thus for some basins, it is possible that

$$SI^*(H^l) = H^l.$$

Now let

$$SI^*(H_f) := \cup_{l \in B_f} SI^*(H^l)$$

and

$$SI^*(H_c) := \cup_{l \in B_c} SI^*(H^l).$$

The subset of states $SI^*(H_f)$ is the sphere of influence of the set of all basins containing defaulted players, while $SI^*(H_c)$ is the sphere of influence of all basins containing no defaulted players

### 3.4 Tipping Points, Systemic, and Very Systemic Players

Again, we begin with a definition.

**Definition 4** (*Endogenous Tipping Points, Systemic and Very Systemic Players*) *Let $\{\omega_t\}_t$ be the state process governed by Markov transition kernel, $p^*(\cdot|\cdot)$, having basins of attraction*

$$\mathcal{A} := \left\{H^1, H^2, \ldots, H^L\right\}.$$

(1) *(Tipping Points) A state $\omega \in \Omega$ is a tipping point if $SR_1^*(\omega, SI^*(H_c)) > 0$ and $SR_1^*(\omega, SI^*(H_f)) > 0$ and*

$$SR_1^*(\omega, SI^*(H_c)) + SR_1^*(\omega, SI^*(H_f)) = 1. \tag{68}$$

*(2) (Systemic and Very Systemic Players) If for tipping point state, $\omega \in \Omega$, with player $i$ not a defaulted player (i.e., with $i \in N \backslash N_f^*(\omega)$) there is a successor state, $\omega' \in \Omega$, such that*

$$N_f^*(\omega') = N_f^*(\omega) \cup \{i\}$$

*and*

$$\omega' \in SI^*(H_f),$$

*then we say that player $i$ is systemic. If there is an $n$-level basin, $H^{l_{rip}}$, containing states in which all players are defaulted, and if player $i$ is such that for some tipping point state, $\omega \in \Omega$, with $i \in N \backslash N_f^*(\omega)$, there is a successor state, $\omega' \in \Omega$, such that*

$$N_f^*(\omega') = N_f^*(\omega) \cup \{i\}$$

*and*

$$\omega' \in SI^*(H^{l_{rip}}),$$

*then we say that player $i$ is very systemic.*

Thus, a very systemic player $i^{rip}$ is a player whose default in a tipping point state propels the state process into the *sphere of influence* of a basin consisting of totally defaulted players. The default of such a player is catastrophic.

We close this section by noting that if each of the finitely many basins of attraction contain states with a defaulted player, then $H_c = \varnothing$, and

$$SR_{[1,\infty)}^*(\omega, H_f) = 1 \text{ for all } \omega \in T.$$

In this case, default by some players (but perhaps not catastrophic default) is inevitable because in finite time with probability 1 the state process will enter a basin of attraction containing defaulted players.

# 4 The Strategic Foundations of Systemic Risk

Our objective now is to construct a game theoretic model of the formation of financial networks by farsighted players who, at the beginning of each period $[t, t+1]$ (at time $t$), after observing the state, $\omega_t$, compute their cash flows, $C_t$, and form a temporary financial network consisting of a short-term investments network and a short-term borrowing-lending-repayment network. Because each player seeks to form financial connections so as to maximize the sum of the expected discounted future payoffs generated by these investment and borrowing-lending connections, we formulate the problem of network formation as a discounted stochastic game. From the stationary Markov equilibrium strategies of the players, we obtain the equilibrium Markov process of network formation as well as the equilibrium state process.

## 4.1 Primitives and Assumptions

A non-cooperative $n$-player, non-zero sum discounted stochastic game (DSG) of network formation is given by the following primitives:

$$\left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{state space}}, \underbrace{(\Phi_i(\omega), U_i(\omega, \cdot, v_i))_{i \in N}}_{\text{one-shot game, } \mathcal{G}(\omega, v)}, \underbrace{q(\cdot|\omega, \cdot)}_{\text{law of motion}} \right\}, \tag{69}$$

where player $i's$ one-shot payoff function is given by

$$U_i(\omega, G^i, G^{-i}, v_i) := (1-\beta_i) r_i(\omega, G^i, G^{-i}) + \beta_i \int_\Omega v_i(\omega') q(\omega'|\omega, G^i, G^{-i}). \tag{70}$$

Now to the details. We will assume $DSG$ satisfies the following list of assumptions:

**[A-4]** *(Discounted Stochastic Games of Financial Network Formation)*
**Feasible Networks:**
*(1) $N$ is a finite set of players consisting of $|N| = n$ players.*
*(2) $D_t$ is the subset of the set of players who are in default during period $[t, t+1]$.*
*(3) $(\Omega, B_\Omega, \mu) = (M^n \times 2^N \times S, B_{M^n} \times 2^{2^N} \times B_S, \lambda \times \eta \times \nu)$ is the state space with typical element $\omega := (C, D, s)$, where $C$ is the cash flow vector, $D$ is the set of defaulted players, and $s$ is the state of the real economy.*
*(4) $A_i = \Delta(Q) \times \mathbb{G}^i \times \mathbb{G}^i$ is the space of network proposals (actions) available to player $i$ with typical element $a_i := (\pi^i, l^{0i}, l^{1i})$ where $\Delta(Q) \times \mathbb{G}^i \times \mathbb{G}^i$ is a compact, convex subset of $R^{m+1} \times R^n \times R^n$, with sum metric*

$$\begin{aligned} \rho_{A_i}(a_i, a_i') &:= \rho_{\Delta(Q) \times \mathbb{G}^i \times \mathbb{G}^i}((\pi^i, l^{0i}, l^{1i}), (\pi'^i, l'^{0i}, l'^{1i})) \\ &:= \left\| \pi^i - \pi'^i \right\|_{R^{m+1}} + \left\| l^{0i} - l'^{0i} \right\|_{R^n} + \left\| l^{1i} - l'^{1i} \right\|_{R^n}. \end{aligned}$$

*For each $\omega \in \Omega$, we will often denote by $G \in \Phi(\omega) := \Phi(C, D, s)$ a typical feasible 3-tuple of network-representing matrices, $(\Pi, L^0, L^1)$, such that for non-defaulted players, $i \in N \backslash D$, $(G)^i \in \mathbb{P}(C^i)$, while for defaulted player's, $i \in D$, $(G)^i = (\pi_0^i, 0, 0)$. Note that a defaulted player has zero investable funds and is constrained to choose network $(\pi_0^i, 0, 0)$, while a non-defaulted player has positive investable funds and also is allowed to choose network $(\pi_0^i, 0, 0)$.*

*(5) $\Phi_i(\cdot)$ is the feasible network proposal correspondence, a measurable set-valued mapping from the state space $\Omega$ into the nonempty, compact, convex subsets of $\Delta(Q)\times\mathbb{G}^i\times\mathbb{G}^i$ given by*

$$\Phi_i(\omega) := \Phi_i(C,D,s) := \begin{cases} \mathbb{P}(C^i) := \Delta(Q)\times\mathbb{B}(C^i)\times\mathbb{G}^i & if \quad i\notin D \\ \{(\pi_0^i,0,0)\} & if \quad i\in D. \end{cases} \tag{71}$$

*Because $\Phi_i(\cdot)$* is *compact-valued and maps from a separable metric space $\Omega$ to a compact metric space $\Delta(Q)\times\mathbb{G}^i\times\mathbb{G}^i$, the measurability of $\Phi_i(\cdot)$ is equivalent to $\Phi_i(\cdot)$ having a measurable graph.*
*Letting $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G} := \prod_{i\in N}[\Delta(Q)\times\mathbb{G}^i\times\mathbb{G}^i]$, equip $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$ with the sum metric,*

$$\rho_{\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}}(G,G') := \sum_i \rho_{\Delta(Q)\times\mathbb{G}^i\times\mathbb{G}^i}((\pi^i,l^{0i},l^{1i}),(\pi'^i,l'^{0i},l'^{1i}))$$

$$:= \sum_{i=1}^n \left(\left\|\pi^i-\pi'^i\right\|_{R^{m+1}} + \left\|l^{0i}-l'^{0i}\right\|_{R^n} + \left\|l^{1i}-l'^{1i}\right\|_{R^n}\right),$$

*a metric compatible with the product topology on $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$. Thus, $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$ is the $\rho_{\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}}$-compact, convex subset of all possible network proposal profiles in $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$ with typical element $G^i := (\pi^i,l^{0i},l^{1i})_{i\in N}\in\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$. Letting*

$$\Phi(\omega) := \Phi(C,D,s) := \Phi_1(C,D,s)\times\cdots\times\Phi_n(C,D,s), \tag{72}$$

*$\Phi(C,D,s)$ is also a measurable set-valued mapping (Lemma 18.4, Aliprantis-Border, 2006) from the state space $\Omega$ into the nonempty, $\rho_{\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}}$-compact, convex subsets of $\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}$. Letting $Gr\Phi(\cdot)$ denote the graph of $\Phi(\cdot)$, we have*

$$Gr\Phi(\cdot) := \left\{(\omega,\Pi,L^0,L^1)\in\Omega\times\Delta(Q)^n\times\mathbb{G}\times\mathbb{G} : (\Pi,L^0,L^1)\in\Phi(\omega)\right\} \tag{73}$$

*with $Gr\Phi(\cdot)\in B_\Omega\times B_{\Delta(Q)^n\times\mathbb{G}\times\mathbb{G}}$.*
**Feasible and Matching Networks and the $\Gamma$-Matching Function:**

*In each cash flow state, $C\in M^n$, we will assume that corresponding to each profile of feasible network proposals, $G\in\mathbb{P}(C)$, the feasible and matching network for the coming period is given by the $\Gamma$-matching function, $\Gamma(\cdot,\cdot)$,*

$$\Gamma(C,G) := \begin{cases} G_0 := (\Pi_0,0,0) & \text{if } G\notin\mathbb{G}(C) \\ G := (\Pi,L^0,L^1) & \text{if } G\in\mathbb{G}(C), \end{cases} \tag{74}$$

*where recall that for all cash flow n-tuples, $C=(C^1,\ldots,C^n)$, in $M^n$,*

$$\mathbb{G}(C) := \Delta(Q)^n\times\mathbb{AM}(C) := \Delta(Q)^n\times(\mathbb{B}(C)\cap\mathbb{M})\times\mathbb{M}.$$

*Let*

$$Gr\mathbb{G}(\cdot) := \{(C,G)\in M^n\times\mathbb{G}^n : G\in\mathbb{G}(C)\},$$

*and*

$$Gr\mathbb{P}(\cdot) := \{(C,G)\in M^n\times\mathbb{G}^n : G\in\mathbb{P}(C)\},$$

*where*

$$\mathbb{P}(C) := \Delta(Q)^n\times\mathbb{B}(C)\times\mathbb{G}.$$

**Payoff Functions and the Law of Motion:**
*(6)* $r_i(\cdot,\cdot)$ *is player* $i'$*s real-valued immediate expected payoff function defined on* $\Omega\times\mathbb{G}$*, such that for each players* $i\in N$ *(i)* $|r_i(\omega,G)|\leq M$ *for all* $(\omega,G)\in\Omega\times\mathbb{G}$*, (ii)* $r_i(\cdot,G)$ *is measurable on* $\Omega$ *for each* $G\in\mathbb{G}$*, and (iii)* $r_i(\omega,\cdot)$ *is continuous on* $\mathbb{G}$ *for each* $\omega\in\Omega$*.*
*(7) The law of motion is given by,*

$$q(d\omega'|\omega,G):=\varepsilon(dC'|D',s')\delta(d(D',s')|\omega,G), \tag{75}$$

*where stochastic kernel,* $(D',s')\longrightarrow\varepsilon(\cdot|D',s')$*, governs cash flows and stochastic kernel,*

$$((C,D,s),G)\longrightarrow\delta(\cdot,\cdot|(C,D,s),G),$$

*governs the set of defaulted players and real economy states. We will assume the following:*
*(a) The law of motion,*

$$(\underbrace{(C,D,s)}_{\omega},G)\longrightarrow q(\cdot|\underbrace{(C,D,s)}_{\omega},G)$$

*is such that for any sequence,* $\{(\omega^n,G^n)\}_n$*, converging to* $(\omega^*,G^*)$*,*

$$q(F|\omega^n,G^n)\longrightarrow q(F|\omega^*,G^*) \tag{76}$$

*for all nonempty* $\rho_\Omega$*-closed subsets* $F$ *of* $\Omega$*.*
*(b) For the stochastic kernel,* $\delta(\cdot,\cdot|\cdot,\cdot)$*, governing the defaulted player sets and real economy states, the following assumptions hold:*
*(i) There exists a product probability measure,* $\eta\times\nu$*, defined on* $(2^N\times S,2^{2^N}\times B_S)$ *such that for all* $(\omega,G)\in\Omega\times\mathbb{G}$ *the probability measure* $\delta(\cdot,\cdot|\omega,G)$ *is absolutely continuous with respect to* $\eta\times\nu$*. Thus,*

$$\delta(\cdot,\cdot|\omega,G)<<\eta\times\nu \text{ for all } (\omega,G)\in\Omega\times\mathbb{G}. \tag{77}$$

*(ii) For all sets* $E\in 2^{2^N}\times B_S$*,* $\delta(E|\cdot,\cdot)$ *is product measurable on* $\Omega\times\mathbb{G}$*.*
*(iii) For each set of defaulted players,* $D$*, the collection of probability density functions on* $S$

$$H_\nu(D):=\{h_D(\cdot|\omega,G):(\omega,G)\in\Omega\times\mathbb{G}\}$$

*of* $\delta(D\times\cdot|\omega,G)$ *with respect to* $v$ *is such that for each state* $\omega\in\Omega$ *the function*

$$G\longrightarrow h_D(s'|\omega,G) \text{ is continuous on } \mathbb{G} \text{ a.e. } [v] \text{ in } s',$$

*and*

$$G^i\longrightarrow h_D(s'|\omega,(G^i,G^{-i})) \text{ is affine on } \mathbb{G}^i \text{ a.e. } [v] \text{ in } s',$$

*(c) For the stochastic kernel,* $\varepsilon(\cdot|\cdot,\cdot)$*, governing cash flow profiles, the following assumptions hold:*
*(i) For all* $(D',s')\in 2^N\times S$ *the probability measure* $\varepsilon(\cdot|D',s')$ *defined on the cash flow state space,* $(M^n,B_{M^n})$*, is absolutely continuous with respect to the nonatomic probability measure* $\lambda$ *defined on* $(M^n,B_{M^n})$*. Thus,*

$$\varepsilon(\cdot|D',s')<<\lambda \text{ for all } (D',s')\in 2^N\times S.$$

*(ii) For all sets* $E\in B_{M^n}$*, the function,* $\varepsilon(E|\cdot,\cdot)$*, is product measurable on* $2^N\times S$*.*
*(iii) The collection of probability density functions on* $M^n$

$$W_\lambda:=\left\{f(\cdot|D',s'):(D',s')\in 2^N\times S\right\}$$

*of* $\varepsilon(\cdot|D',s')$ *with respect to* $\lambda$ *is such that*

$$(D',s')\longrightarrow f(C'|D',s')$$

*is measurable on* $2^N\times S$ *a.e.* $[\lambda]$ *in* $C'$*.*

## 4.2 Comments on the Primitives and Assumptions

### 4.2.1 Player Value Functions

The key ingredient in analyzing any discounted stochastic game (henceforth, DSG) is the DSG's parameterized collection of *one-shot games*,

$$\mathcal{G}(\omega, v)_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}. \tag{78}$$

The importance of this parameterized collection derives from the Theorem of Blackwell (1965) giving necessary and sufficient conditions for the existence of stationary Markov equilibria in terms the equilibria of this collection of one-shot games. In due course we will discuss Blackwell's Theorem in more detail. For now we focus upon the space of valuation function profiles,

$$v := (v_1, \ldots, v_n) \in \mathcal{L}_X^\infty, \tag{79}$$

which together with states $\omega \in \Omega$ indexes our collection of one-shot games.

Players in a discounted stochastic game are guided in making their strategy choices by state-contingent prices or values. For each player $i$, this vector of state-contingent values is given by a function, $v_i : \Omega \longrightarrow R$. Taken together, players' state-contingent value functions form an $n$-tuple of functions which we will refer to as players' value function profile, $v := (v_1, \ldots, v_n)$. As in the literature on discounted stochastic games (e.g., see Nowak and Raghavan, 1992), the space of players' value function profiles is given by

$$\mathcal{L}_X^\infty := \mathcal{L}_{X_1}^\infty \times \cdots \times \mathcal{L}_{X_n}^\infty, \tag{80}$$

where for each player $i = 1, 2, \ldots, n$, $\mathcal{L}_{X_i}^\infty$ is space of $\mu$-equivalence classes of functions, $v_i : \Omega \longrightarrow R$, such that $v_i(\omega) \in X_i$ a.e. $[\mu]$. For each player $i$, $X_i$ is the closed bounded interval, $[-M, M]$, the same for each player. Players' payoffs (both immediate and discounted) reside in the closed, bounded, convex subset, $X := X_1 \times \cdots \times X_n = [-M, M]^n$, and thus, players' value function profiles reside in the space, $\mathcal{L}_X^\infty$, a metrizable, weak star compact, convex subset of $\mathcal{L}_{R^n}^\infty$.

Formally, let $\mathcal{L}_R^1(\mu, \Omega) := \mathcal{L}_R^1$ denote the separable Banach space of $\mu$-equivalence classes of $\mu$-integrable functions, $u : \Omega \longrightarrow R$ with norm

$$\|u\|_1 := \int_\Omega |u|\, d\mu.$$

Also, denote by $L_R^1$ the prequotient of $\mathcal{L}_R^1$ (i.e., the space of all real-valued, integrable functions), and let

$$\mathcal{L}_{R^n}^1 := \underbrace{\mathcal{L}_R^1 \times \cdots \times \mathcal{L}_R^1}_{n \text{ times}}$$

denote the separable Banach space of $\mu$-equivalence classes of $\mu$-integrable functions, $U : \Omega \longrightarrow R^n$, $U := (U_1, \ldots U_i, \ldots, U_n)$, with norm

$$\|U\|_1 = \sum_{i=1}^n \|U_i\|_1.$$

Next, let $\mathcal{L}_R^\infty$ denote the Banach space of $\mu$-equivalence classes of $\mu$-essentially bounded functions, $v : \Omega \longrightarrow R$ with norm

$$\|v\|_\infty := ess\sup v := \inf\{x \in R : \mu\{\omega : |v(\omega)| > x\} = 0\}. \tag{81}$$

$\mathcal{L}_R^\infty$ is the norm dual of $\mathcal{L}_R^1$. Equip $\mathcal{L}_R^\infty$ with the weak star topology, denoted by $w^*$ or $\sigma(\mathcal{L}_R^\infty, \mathcal{L}_R^1)$. We will denote by $L_R^\infty$ the prequotient of $\mathcal{L}_R^\infty$ (i.e., the space of all real-valued, $\mu$-essentially bounded functions).

For $i = 1, 2, \ldots, n$, let $X_i$ be the closed bounded interval $[-M, M] \subset R$, and let

$$\mathcal{L}^{\infty}_{X_i} := \{v \in \mathcal{L}^{\infty}_R : v(\omega) \in X_i \text{ a.e. } [\mu]\}. \tag{82}$$

Equip $\mathcal{L}^{\infty}_{X_i}$ with the compact and metrizable relative weak star topology, denoted by $w_i^*$ or $\sigma(\mathcal{L}^{\infty}_{X_i}, \mathcal{L}^{1}_{X_i})$.[5] To fix the metric and hence the notation, let $\rho_{w_i^*}$ be the metric on $\mathcal{L}^{\infty}_{X_i}$ compatible with the weak star topology. Also, let $\rho_{X_i}$ denote the metric on $X_i$ where for $x$ and $x'$ in $X_i$, $\rho_{X_i}(x, x') := |x - x'|$.

Finally, let $X := X_1 \times \cdots \times X_n$ and consider the Cartesian product,

$$\mathcal{L}^{\infty}_{X} := \mathcal{L}^{\infty}_{X_1} \times \cdots \times \mathcal{L}^{\infty}_{X_n}, \tag{83}$$

equipped with the the sum metric,

$$\rho_{w^*} := \textstyle\sum_{i=1}^{n} \rho_{w_i^*}, \tag{84}$$

a metric compatible with the relative weak star product topology, $w^*$, on $\mathcal{L}^{\infty}_{X}$, and equip $X$ with the sum metric

$$\rho_X := \textstyle\sum_{i=1}^{n} \rho_{X_i}. \tag{85}$$

### 4.2.2 Continuity Properties

We begin with some fundamental results on the continuity properties the law of motion and players' payoff functions. For each $\omega \in \Omega \times \mathcal{L}^{\infty}_{X}$, consider the $X$-valued function,

$$(G, v) \longrightarrow U(\omega, G, v) := (U_1(\omega, G, v_1), \ldots, U_n(\omega, G, v_n)),$$

defined on $\mathbb{G}(C) \times \mathcal{L}^{\infty}_{X}$ taking values in $X \subset R^n$. For each player $i = 1, \ldots, n$,

$$U_i(\omega, G, v_i) := (1 - \beta_i) r_i(\omega, G) + \beta_i \int_{\Omega} v_i(\omega') q(d\omega' | \omega, G) \tag{86}$$

(1) By part (6) of [A-4]. we have via Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that

$$\left.\begin{array}{c} G^n \underset{\rho_{\Delta(Q)^n \times \mathbb{G} \times \mathbb{G}}}{\longrightarrow} G^* \\ \text{implies that} \\ \sup_{E \in B_\Omega} |q(E|\omega, G^n) - q(E|\omega, G^*)|_R \longrightarrow 0, \end{array}\right\} \tag{87}$$

sometimes written $\|q(\cdot|\omega, G^n)) - q(\cdot|\omega, G^*)\|_\infty \longrightarrow 0$.

(2) Under parts (6) and (7) of [A-4] for each cash flow state, $C \in M^n$, each player's expected payoff function, $(G, v_i) \longrightarrow U_i(\omega, G, v_i) \in X_i$, is $\rho_{\mathbb{G}(C) \times w_i^*}$-continuous in $(G, v_i) \in \mathbb{G}(C) \times \mathcal{L}^{\infty}_{X_i}$ - so that in each cash flow state, $C \in M^n$, the $X$-valued function,

$$(G, v) \longrightarrow U(\omega, G, v) \in X,$$

is $\rho_{\mathbb{G}(C) \times w^*}$-continuous in $(G, v) \in \mathbb{G}(C) \times \mathcal{L}^{\infty}_{X}$. In fact, we can say more about the collection of functions, $U(\omega, \cdot, v) : \mathbb{G}(C) \longrightarrow X$, for $(\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{X}$. In particular, as has been shown by Salon (1998), *for each state $\omega \in \Omega$ the collection of functions,*

$$\{U(\omega, \cdot, v) : v \in \mathcal{L}^{\infty}_{X}\}, \tag{88}$$

[5]Because the Borel $\sigma$-field $B_\Omega$ is countably generated, the space of $\mu$-equivalence classes of $\mu$-integrable functions, $\mathcal{L}^{1}_{R}$, is separable. As a consequence, the set of value function $\mu$-equivalence classes $\mathcal{L}^{\infty}_{X_i}$ is a compact, convex, and metrizable subset of $\mathcal{L}^{\infty}_{R}$ for the weak star topology (e.g., see Nowak and Raghavan, 1992).

*is uniformly equicontinuous on* $\mathbb{G}(C)$.[6] To see this, let

$$U_{\omega v_i}(\cdot) := (1-\beta_i) r_i(\omega, \cdot) + \beta_i \int_\Omega v_i(\omega') q(d\omega'|\omega, \cdot).$$

For fixed $\omega$, we have for each $v \in \mathcal{L}_X^\infty$

$$\begin{aligned} &|U_{\omega v_i}(G) - U_{\omega v_i}(G')| \\ &\leq (1-\beta_i) |r_i(\omega, G) - r_i(\omega, G')| \\ &+\beta_i M \left| \int_\Omega q(d\omega'|\omega, G) - \int_\Omega q(d\omega'|\omega, G') \right|. \end{aligned}$$

Because

$$r_i(\omega, \cdot) \text{ and } H_\omega(\cdot) := \int_\Omega q(d\omega'|\omega, \cdot)$$

are continuous functions on a compact set, and hence uniformly continuous, for any $\frac{\varepsilon}{2} > 0$ there is a $\delta > 0$ such that for any $G$ and $G'$ in $\mathbb{G}(C)$ with $\rho_{\mathbb{G}(C)}(G, G') < \delta$

$$|r_i(\omega, G) - r_i(\omega, G')| < \tfrac{\varepsilon}{2}$$
and
$$|H_\omega(G) - H_\omega(G')| < \tfrac{\varepsilon}{2}.$$

### 4.2.3 The Default and Matching Adjusted Cash Flow Transition Function

By the Corollary in Rao and Rao (1972), because $\lambda$ is nonatomic, $\lambda \times \eta \times v$ is nonatomic (see [A-4] (b)(i) and (c)(i).[7] Because project portfolios are formed at the beginning of the period and project returns are realized at the end of the period - and because borrowing and lending contracts are proposed at the beginning of the period and proposed repayments are made at the end of the period, each player's payoff is an expected proposed payoff. In order for a player's expectations to be realized, loanable funds contracts must be matching. In particular, if at the beginning of the period players' propose a feasible network given by $G := (\pi^i, l^{0i}, l^{1i})_{i \in N}$, where

$$G := (\Pi, L^0, L^1) \in \Phi(C, D, s) := \prod_{i=1}^n \left\{ \left[ \Delta(Q) \times \mathbb{B}(C^i) \times \mathbb{G}^i \right] (1 - I_D(i)) + [\pi_0^i, 0, 0] I_D(i) \right\}$$

so that for each $i$

$$G^i := (\pi^i, l^{0i}, l^{1i}) \in \Phi_i(C, D, s) := [\Delta(Q) \times \mathbb{B}(C^i) \times \mathbb{G}^i](1 - I_D(i)) + [\pi_0^i, 0, 0] I_D(i),$$

then if players' expectations are to be realized, players' network proposal must also be matching. Thus, we must have $G \in \mathbb{G}(C)$. Then each player's end-of-period, state-contingent cash flow is given by,

$$C_{t+1}^i := \left\langle r_{t+1}, \pi^i \right\rangle \left( [C_t^i - \left\langle l^{0i}(\omega_t), e \right\rangle ] \vee 0 \right) + \left\langle p(r_{t+1}, C_t, G_t), (m_i - e_i) \right\rangle.$$

[6] The collection,

$$\{U(\omega, \cdot, v) : v \in \mathcal{L}_X^\infty\},$$

is uniformly equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $G$ and $G'$ in $\mathbb{G}(C)$ with $\rho_{\mathbb{G}(C)}(G, G') < \delta$,

$$\rho_X(U(\omega, G, v), U(\omega, G', v)) < \varepsilon,$$

for all $v \in \mathcal{L}_X^\infty$.

[7] $E \subset M^n$ is an atom of $M^n$ relative to $\lambda(\cdot)$ if the following implication holds: if $\lambda(E) > 0$, then $H \subset E$ implies that $\lambda(H) = 0$ or $\lambda(E - H) = 0$. If $M^n$ contains no atoms relative to $\lambda(\cdot)$, $M^n$ is said to be atomless or nonatomic. Because $M^n$, is a complete, separable metric space (as a closed subset of $R^n$) $\lambda(\cdot)$ is atomless (or nonatomic) if and only if $\lambda(\{C\}) = 0$ for all $C \in M^n$ (see Hildenbrand, 1974, pp 44-45).

The vector of players' end-of-period, state-contingent cash flows is given by the vector

$$C_{t+1} := r_{t+1}\Pi^T I \left([C_t - L^0(\omega_t)e] \vee 0\right) + p(r_{t+1}, C_t, G_t)\left[M^T - I\right].$$

Adopting the $\Gamma$-matching function (which, for each current state, $\omega_t$, maps feasible network proposals to feasible and matching networks), (realizable) end-of-period, state-contingent cash flows are given by the vector

$$\left.\begin{array}{c} C_{t+1} \\ := r_{t+1}\Gamma^{\pi}(C_t, G_t)^T I \left([C_t - \Gamma^0(C_t, G_t)e] \vee 0\right) + \\ p^i(r_{t+1}, C_t, \Gamma(C_t, G_t))\left[M^T - I\right], \end{array}\right\} \tag{89}$$

where $\Gamma^{\pi}(C_t, G_t) = \Pi(C_t)$ is the investments network, $\Gamma^0(C_t, G_t) = L^0(C_t)$ is the feasible and matching borrowing-lending network, and $\Gamma^1(C_t, G_t) = L^1(C_t)$ is the corresponding matching repayment network.[8] Thus, under the $\Gamma$-matching function, the (realizable) expected payoff to player $i$ under any profile of feasible network proposals is given by

$$U_i(\omega, \Gamma(C, G), v_i) := (1 - \beta_i) r_i(\omega, \Gamma(C, G)) + \beta_i \int_{\Omega} v_i(\omega') q(\omega' | \omega, \Gamma(C, G)) \tag{90}$$

At any time $t$ cash flows at $t + 1, C_{t+1}$, are a function of current cash flows, $C_t$, players's current actions, $\Gamma(C_t, G_t) := (\Pi_t, L_t^0, L_t^1)$, the stationary default adjustment function, $p(\cdot, \cdot, \cdot)$, and risky project returns at $t+1$, $r_{t+1}$. Thus, rewriting expression (43) in abstract vector function form we have

$$C_{t+1} = F_p(C_t, \Gamma(C_t, G_t), r_{t+1}). \tag{91}$$

If the current state is $\omega_t := (C_t, D_t, s_t)$, then player $i's$ immediate expected payoff is

$$r_i(\omega_t, \Gamma(C_t, G_t)) := \int_{\mathcal{R}} u_i(F_p(C_t, \Gamma(C_t, G_t), r_{t+1})) g(dr_{t+1} | D_t, s_t), \tag{92}$$

where

$$\left\{g(dr_{t+1} | D_t, s_t) : (D_t, s_t) \in 2^N \times S\right\}$$

is the collection of product measurable probability density functions describing the random behavior of project returns as a function of the state of the real economy and the set of defaulted players.

We close this section with a critical observation. If at time $t$ in state $\omega_t$, player $i$ qualifies for the set of permanently defaulted players (i.e., if $\omega_t \in \Omega_f(i)$ or $i \in N_f(\omega_t)$), then given the nature of the default and matching adjusted cash flow transition function for player $i$,

$$C_{t+1}^i := \left\langle r_{t+1}, \pi_t^i \right\rangle \left([C_t^i - \left\langle l^{0i}(\omega_t), e \right\rangle] \vee 0\right) + \left\langle p(r_{t+1}, C_t, G_t), (m_i - e_i) \right\rangle,$$

as well as player $i's$ feasible constraint correspondence,

$$\Phi_i(\omega) := \Phi_i(C, D, s) := \begin{cases} \mathbb{P}(C^i) := \Delta(Q) \times \mathbb{B}(C^i) \times \mathbb{G}^i & if \quad i \notin D \\ \{(\pi_0^i, 0, 0)\} & if \quad i \in D, \end{cases}$$

[8]Thus, we have that, if the affordable proposed network $G = (\Pi, L^0, L^1)$ is not matching, then $\Gamma^{\pi}(C, G) = \Gamma^{\pi}(C, \Pi, L^0, L^1) = \Pi_0$ - and if $G$ is matching, then $\Gamma^{\pi}(C, \Pi, L^0, L^1) = \Pi$. Also $\Gamma^0(C, \Pi, L^0, L^1) = 0$ if $G = (\Pi, L^0, L^1)$ is not matching and $\Gamma^0(C, \Pi, L^0, L^1) = L^0$if $G = (\Pi, L^0, L^1)$ is matching. And finally, $\Gamma^1(C, \Pi, L^0, L^1) = 0$ if $G = (\Pi, L^0, L^1)$ is not matching and $\Gamma^1(C, \Pi, L^0, L^1) = L^1$if $G = (\Pi, L^0, L^1)$ is matching.

it follows that player $i$ will be a member of $D(\omega_{t+1}) := proj_{2^N}(\omega_{t+1}) = D_{t+1}$ during the period $[t+1, t+2]$, and as a consequence, will re-qualify for membership in the set of defaulted players, $D_{t+2}$, for period $[t+2, t+3]$. This is because for player $i \in D_{t+1}$, $\Phi_i(C_{t+1}, D_{t+1}, s_{t+1}) = \{(\pi_0^i, 0, 0)\}$, implying that $l^{0i}(\omega_{t+1}) = 0$, and because

$$\omega_t \in \Omega_f(i) := \left\{\omega \in \Omega : C^i(\omega) \leq 0 \text{ and } C^i(\omega) - \left\langle l^{0i}(\omega), e\right\rangle \leq 0\right\},$$

we have

$$C^i_{t+1} := \left\langle r_{t+1}, \pi^i\right\rangle \underbrace{([C^i_t - \left\langle l^{0i}(\omega_t), e\right\rangle] \vee 0)}_{0} + \left\langle \underbrace{p(r_{t+1}, C_t, G_t)}_{0}, (m_i - e_i)\right\rangle = 0,$$

implying that $\omega_{t+1} \in \Omega_f(i)$ - in turn implying that $i \in D_{t+2}$ (recall that $C^i_t := C^i(\omega_t)$, and thus, $C_t := (C^1(\omega_t), \ldots, C^n(\omega_t))$ and $C^i_{t+1} := C^i(\omega_{t+1})$). Letting

$$P_D(N) := \{D' \in 2^N : D \subseteq D'\}$$

(i.e., the collection of all subsets of players containing the subset $D$), we can conclude from the above observations that for all $(\omega, G) := ((C, D, s), G) \in \Omega \times \mathbb{G}$,

$$\delta(P_D(N) \times S|\omega, G) = 1. \tag{93}$$

## 4.3 Pareto Optimal Matching and Pure Strategy Nash Equilibria

Under the our matching function, pure strategy stationary Markov equilibria are easy to find - we need only consider feasible and matching networks having payoffs greater than the payoffs to the zero network $G_0$ for all players. Here we will focus on the much smaller set of Pareto optimal pure Nash equilibria. To begin, let $(\omega, v) \in \Omega \times \mathcal{L}^\infty_X$ be given and consider the one-shot game of financial network formation,

$$\mathcal{G}(\Gamma(C, \cdot), v) := \{\Phi_i(\omega), U_i(\omega, \Gamma(C, \cdot), v_i)\}_{i \in N}, \tag{94}$$

with matching function, $\Gamma(C, \cdot)$. Under the $\Gamma$-matching function, $\Gamma(C, \cdot)$, for each $\omega$ we have

$$U_i(\omega, \Gamma(C, \mathbb{P}(C)), v_i) = U_i(\omega, \mathbb{G}(C), v_i)$$

and for all unmatched feasible networks, $G \in \mathbb{P}(C(\omega)) \backslash \mathbb{G}(C(\omega))$, we have

$$U_i(\omega, \Gamma(C, G), v_i) = U_i(\omega, G_0, v_i).$$

**Definition 2** *(Feasible, Pareto Optimal Matching Networks)*
*Given $(\omega, v) \in \Omega \times \mathcal{L}^\infty_X$, let $G \in \Phi(\omega)$be given. We say that feasible, matching network $G$ is Pareto dominated by feasible, matching network $G' \in \Phi(\omega)$, if*

$$\begin{gathered} U_i(\omega, G', v_i) \geq U_i(\omega, G, v_i) \text{ for all } i \\ \text{and} \\ U_{i'}(\omega, G', v_{i'}) > U_{i'}(\omega, G, v_{i'}) \text{ for some } i' \end{gathered}$$

*Given $v \in \mathcal{L}^\infty_X$, we say that $G^* \in \Phi(\omega)$ is Pareto optimal if it is not Pareto dominated by any other feasible, matching network $G' \in \Phi(\omega)$. We will denote by $\mathcal{PO}(\omega, v)$ the set of all Pareto optimal feasible and matching networks in $\Phi(\omega)$ corresponding to the one-shot game, $\mathcal{G}(\Gamma(C, \cdot), v)$.*

Denote by

$$\left.\begin{array}{c} P_{(\omega,v,\varepsilon)}(G_0) \\ := \{G \in \Phi(\omega) : U_i(\omega, \Gamma(C,G), v_i) \geq U_i(\omega, G_0, v_i) + \varepsilon \text{ for all } i\}, \end{array}\right\} \quad (95)$$

the collection of all feasible network proposals with expected payoff at least $\varepsilon$ greater than the expected payoff of the zero network $G_0$ for all players, $i$. We will assume the following:

**[A-5]** *(The Zero Network, $G_0 \in \Phi(\omega)$, is Pareto Dominated for Each $(\omega, v) \in \Omega \times \mathcal{L}_X^\infty$) Given a $\mathcal{DSG}$ of financial network formation satisfying [A-4], with collection of one-shot games, $\{\mathcal{G}(\Gamma(C,\cdot), v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty}$, there exists an $\varepsilon > 0$ such that for each one-shot network formation game in the collection,*

$$\{\mathcal{G}(\Gamma(C,\cdot), v)\}_{(\omega,v)\in\Omega\times\mathcal{L}_X^\infty},$$

*with non-defaulted players, there exists a feasible network, $G_{(\omega,v,\varepsilon)} \in \Phi(\omega) := \Phi(C, D, s)$, $D \neq N$, such that*

$$U_i(\omega, \Gamma(C, G_{(\omega,v,\varepsilon)}), v_i) \geq U_i(\omega, G_0, v_i) + \varepsilon \text{ for all } i \in N \backslash D.$$

Note that under the matching function, $\Gamma$, all the networks contained in $P_{(\omega,v,\varepsilon)}(G_0)$ are not only feasible, but also matching. Hence, $P_{(\omega,v,\varepsilon)}(G_0) \subset \mathbb{G}(C)$. Moreover, all the networks in $P_{(\omega,v,\varepsilon)}(G_0)$ are Nash. To see this, consider network

$$G = (\pi^i, l^i)_{i\in N} = (\pi^i, l^{0i}, l^{1i})_{i\in N} = ([\pi_{iq}]_{iq\in N\times Q}, [l^0_{ij}]_{ij\in N^2}, [l^1_{ij}]_{ij\in N^2}) \in P_{(\omega,v,\varepsilon)}(G_0).$$

Suppose player $i$ defects from $l^i$ to $l'^i$. Because the loanable funds network, $(l^i)_{i\in N}$, is matching, we have, $l^t_{ij} + l^t_{ji} = 0$ for all $ij \in N^2$ and $t = 0$ or 1, while after the defection, $l^{t\prime}_{ij} + l^t_{ji} \neq 0$, for some $t = 0$ or 1, and some counter party $j \in N$. Thus $G' \notin \mathbb{G}(C)$, implying that $\Gamma(C, G') = G_0$. Therefore, for the defector (who started out in network $G$) $U_i(\omega, \Gamma(C,G), v_i) > U_i(\omega, \Gamma(C,G'), v_i) = U_i(\omega, G_0, v_i)$. Therefore, $G \in P_{(\omega,v,\varepsilon)}(G_0)$ is a Nash equilibrium.

Consider the set-valued mapping,

$$(\omega, v) \longrightarrow P_{(\omega,v,\varepsilon)}(G_0). \quad (96)$$

Under assumptions [A-1]-[A-5] it is easy to show that the correspondence, $(\omega, v) \longrightarrow P_{(\omega,v,\varepsilon)}(G_0)$, is Caratheodory - measurable in $\omega$ and continuous in $v$.

Letting $\mathcal{N}(\omega, v)$ denote the set of Nash equilibria of the one-shot game, $\{\mathcal{G}(\Gamma(C,\cdot), v)\}$, we have for each $(\omega, v) \in \Omega \times \mathcal{L}_X^\infty$

$$P_{(\omega,v,\varepsilon)}(G_0) \subset \mathcal{N}(\omega, v) \subset \mathbb{G}(C). \quad (97)$$

Next consider the Pareto problem

$$\max_{G\in\Phi(\omega)} \sum_i^n U_i(\omega, \Gamma(C,G), v_i) \quad (98)$$

Under assumptions [A-1]-[A-5]

$$\max_{G\in\Phi(\omega)} \sum_i^n U_i(\omega, \Gamma(C,G), v_i) = \max_{G\in P_{(\omega,v,\varepsilon)}(G_0)} \sum_i^n U_i(\omega, G, v_i). \quad (99)$$

Let

$$U^{po}(\omega,\Gamma(C,\Phi(\omega)),v) := \max_{G\in\Phi(\omega)} \sum_i^n U_i(\omega,\Gamma(C,G),v_i)$$

and

$$\mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0) = \left\{ G \in \Phi(\omega) : \sum_i^n U_i(\omega,\Gamma(C,G),v_i) \geq U^{po}(\omega,\Gamma(C,\Phi(\omega)),v) \right\}. \quad (100)$$

By the measurable version of Berge Maximum Theorem (see Aliprantis and Border, 2006), we have that the Pareto function, , $(\omega,v) \longrightarrow U^{po}(\omega,\Gamma(C,\Phi(\omega)),v)$, is Caratheodory - measurable in $\omega$ and continuous in $v$. Moreover, because the set-valued mapping $(\omega,v) \longrightarrow P_{(\omega,v,\varepsilon)}(G_0)$ is Caratheodory, the Pareto optimal network correspondence, $(\omega,v) \longrightarrow \mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0)$, is upper Caratheodory - jointly measurable in $\omega$ and $v$, and upper semicontinuous in $v$. Thus, in each state, $\omega$, $\mathcal{N}^*_{(\omega,\cdot,\varepsilon)}(G_0)$ is a sub-USCO contained in the $\omega$-Nash USCO, $\mathcal{N}(\omega,\cdot)$. We summarize all of this in the following result.

**Theorem 2** *(The Pareto USCO)*
*Suppose assumptions [A-1]-[A-5] hold. Then the collection of one-shot games, $\{\mathcal{G}(\Gamma(C,\cdot),v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^\infty_X}$, possesses an upper Caratheodory, Pareto optimal Nash network correspondence, $(\omega,v) \longrightarrow \mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0)$, with*

$$\mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0) \subset P^*_{(\omega,v,\varepsilon)}(G_0) \subset \mathcal{N}^*(\omega,v) \subset \mathbb{G}(C), \text{ for each } (\omega,v) \in \Omega\times\mathcal{L}^\infty_X,$$

*and USCO part, $v \longrightarrow \mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0)$.*

# 5 Stationary Markov Equilibrium in Network Formation Strategies

Corresponding to the Pareto optimal Nash network correspondence, $\mathcal{N}^*_{(\cdot,\cdot,\varepsilon)}(G_0)$, there is a Pareto optimal Nash payoff correspondence, $\mathcal{P}^*_{(\cdot,\cdot,\varepsilon)}(G_0)$, where for each $(\omega,v) \in \Omega\times\mathcal{L}^\infty_X$, $\mathcal{P}^*_{(\omega,v,\varepsilon)}(G_0)$ is given by

$$\left.\begin{aligned} \mathcal{P}^*_{(\omega,v,\varepsilon)}(G_0) &:= \left\{ U \in M^n : U = U(\omega,\Gamma(C,G),v) \text{ for some } G \in \mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0) \right\} \\ &:= U(\omega,\mathcal{N}^*_{(\omega,v,\varepsilon)}(G_0),v). \end{aligned}\right\} \quad (101)$$

Like $\mathcal{N}^*_{(\cdot,\cdot,\varepsilon)}(G_0)$, the correspondence, $\mathcal{P}^*_{(\cdot,\cdot,\varepsilon)}(G_0)$ is Caratheodory, but taking nonempty, closed values in $X$. For each value function profile, $v \in \mathcal{L}^\infty_X$, let

$$\mathcal{S}^\infty(\mathcal{P}^*_{(\cdot,v,\varepsilon)}(G_0)) := \mathcal{S}^\infty(U(\cdot,\mathcal{N}^*_{(\cdot,v,\varepsilon)}(G_0),v)) \quad (102)$$

denote the collection of all $\lambda\times\eta\times v$-equivalence classes of measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}^*_{(\omega,v,\varepsilon)}(G_0)$. According to the version of Blackwell's Theorem (1965) given in Page (2015), the discounted stochastic game of financial network formation specified in assumptions [A-1]-[A-5] has a stationary Markov equilibrium in network formation strategies if and only if the Nash payoff selection correspondence, in this case the correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}^*_{(\cdot,v,\varepsilon)}(G_0)),$$

has a fixed point.[9] Moreover, according to results in Page (2015), the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}^{*}_{(\cdot,v,\varepsilon)}(G_0))$, will have a fixed point provided it is a $K$-correspondence. Because our discounted stochastic game of financial network formation game is risky, the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}^{*}_{(\cdot,v,\varepsilon)}(G_0))$, is a $K$-correspondence. Therefore, by Page (2015), there exists a profile of value function-stationary network strategy pairs, $(v^*(\cdot), G^*(\cdot)) \in \mathcal{S}^{\infty}(\mathcal{P}^{*}_{(\cdot,v^*,\varepsilon)}(G_0)) \times \mathcal{S}^{\infty}(\mathcal{N}^{*}_{(\cdot,v^*,\varepsilon)}(G_0))$, where

$$\omega \longrightarrow v^*(\omega) = U(\omega, G^*(\omega), v) \in \mathcal{P}^{*}_{(\omega,v^*,\varepsilon)}(G_0) \text{ a.e.}[\mu], \tag{103}$$

and

$$\omega \longrightarrow G^*(\omega) := ([\pi^*_{iq}(\omega)]_{iq}, [l^{0*}_{ij}(\omega)]_{ij}, [l^{1^*}_{ij}(\omega)])_{i\in N} \in \mathcal{N}^{*}_{(\omega,v,\varepsilon)}(G_0) \text{ a.e.}[\mu]. \tag{104}$$

Our main result on existence is the following (see Page 2015 for details and a proof).

**Theorem 3** *(Existence of Stationary Pareto Optimal Markov Equilibrium)*
*Suppose assumptions [A-1]-[A-5] hold. Then for the discounted stochastic games of financial network formation,*

$$\left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{state space}}, \underbrace{(\Phi_i(\omega)), U_i(\omega, \Gamma(C,\cdot), v_i))_{i\in N}}_{\text{one-shot game, } \mathcal{G}(\Gamma(C,\cdot),v)}, \underbrace{q(\cdot|\omega, \Gamma(C,\cdot))}_{\text{law of motion}} \right\}$$

*with collection of one-shot games, $\{\mathcal{G}(\Gamma(C,\cdot), v)\}_{(\omega,v)\in\Omega\times\mathcal{L}^{\infty}_X}$, and $\Gamma$-matching function,*

$$(C, G) \longrightarrow \Gamma(C, G) := \begin{cases} G_0 := (\Pi_0, 0, 0) & \text{if } G \notin \mathbb{G}(C) \\ G := (\Pi, L^0, L^1) & \text{if } G \in \mathbb{G}(C), \end{cases}$$

*there exists a profile of value functions, $v^* \in \mathcal{L}^{\infty}_X$, and a profile of stationary Markov network formation strategies, $\omega \longrightarrow G^*(\omega)$, forming a Nash equilibrium in space of all profiles of stationary Markov network proposal strategies.*

We have, without loss of generality, $G^*(\omega) \in \mathcal{N}^{*}_{(\omega,v,\varepsilon)}(G_0)$ for all $\omega$ - implying that

$$G^*(\omega) \in \mathbb{G}(C(\omega)) \text{ for all } \omega,$$

where $C(\omega) := proj_{M^n}(\omega) = C \in M^n$.

# 6 Stability Properties of the Dynamics of Temporary Financial Networks

## 6.1 Absorbing Sets and Invariant and Ergodic Probability Measures

A set of states, $E \in B_\Omega$, is called a $p^*$-absorbing if $p^*(E|\omega) = 1$ for all states, $\omega = (C, D, s) \in E$. Let $\mathcal{L}^* \subseteq B_\Omega$ denote the collection of all $p^*$-absorbing sets. A $p^*$-absorbing

[9] It is important to note that by Blackwell's Theorem (1965) for each player $i$, in equilibrium, player $i$'s stationary Markov strategy is optimal against defections, not only to other stationary Markov strategies, but also to all other strategies including history dependent strategies - provided the other players continue to play their stationary Markov strategies. Thus, the stationary Markov equilibrium, whose existence is established here, is truly a Nash equilibrium in strategies. This is an often forgotten part of Blackwell's seminal 1965 result.

set $E \in \mathcal{L}^*$ is said to be *indecomposable* if it does not contain the union of two disjoint absorbing sets. Note that the set of all absorbing sets is closed under countable unions and intersections.

A probability measure $\gamma(\cdot)$ on the state space $(\Omega, B_\Omega)$ is invariant for Markov transition $p^*(\cdot|\cdot)$ (i.e., is $p^*$-invariant) if

$$\gamma(E) = \int_\Omega p^*(E|\omega)d\gamma(\omega) \text{ for all } E \in B_\Omega. \tag{105}$$

Thus, if probability measure $\gamma(\cdot)$ is $p^*$-invariant, then for any set of states $E \in B_\Omega$, if the current status quo state $\omega_t = (C_t, D_t, s_t)$ is chosen according to probability measure $\gamma(\cdot)$ - so that the probability that $\omega_t$ lies in $E$ is $\gamma(E)$ - then the probability that next period's state $\omega_{t+1} = (C_{t+1}, D_{t+1}, s_{t+1})$ lies in $E$ is also $\gamma(E) = \int_\Omega p^*(E|\omega)d\gamma(\omega)$. Denote by $\mathcal{I}^*$ the collection of all $p^*$-invariant measure.

A $p^*$-invariant measure $\gamma(\cdot)$ is said to be $p^*$-ergodic if $\gamma(E) = 0$ or $\gamma(E) = 1$ for all $E \in \mathcal{L}^*$. Denote by $\mathcal{E}^*$ the collection of all $p^*$-ergodic measures. Because the $p^*$-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{I}^*$ of $p^*$-invariant measures (see Theorem 19.25 in Aliprantis and Border 2006), each measure $\gamma(\cdot)$ in $\mathcal{I}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.

## 6.2 Visitations Times

The number of visitations by the state process $\{\omega_t\}_t$ to the set of states $E \in B_\Omega$, is given by

$$\eta_E := \sum_{t=1}^{\infty} I_E(\omega_t), \tag{106}$$

where $I_E(\omega_t) = 1$ if $\omega_t \in E$ and zero otherwise. Thus, the expected number of visitations to $E$ starting from state $\omega = (C, D, s)$ is given by

$$G^*(\omega, E) := E_\omega[\eta_E] = \sum_{t=1}^{\infty} p^{*t}(E|\omega). \tag{107}$$

The probability that the state process $\{\omega_t\}_t$ visits $E$ infinitely often (denoted by i.o.) is given by

$$\left.\begin{aligned} Q(\omega, E) &:= \mu\{\omega_t \in E \text{ i.o.}|\omega_0 = \omega\} = \mu\{\eta_E = \infty|\omega_0 = \omega\} \\ &= \mu\{\cap_{t'=1}^{\infty} \cup_{t=t'}^{\infty} (\omega_t \in E|\omega_0 = \omega)\} \text{ for all } \omega \in \Omega. \end{aligned}\right\} \tag{108}$$

By Proposition 9.1.1 in Meyn and Tweedie (2009), if for any $E \in B_\Omega$, $L(\omega, E) = 1$ for all $\omega \in E$, then

$$L(\omega, E) := \mu\{\tau_E < \infty|\omega_0 = \omega\} = Q(\omega, E) \text{ for all } \omega \in \Omega. \tag{109}$$

## 6.3 Recurrence, Transience, and Irreducibility

The set of states $E$ is *recurrent* if

$$G^*(\omega, E) := E_\omega[\eta_E] = \sum_{t=1}^{\infty} p^{*t}(E|\omega) = +\infty.$$

By Proposition 8.1.3 in Meyn and Tweedie (2009), for any state $\omega \in \Omega$,

$$G^*(\omega, \{\omega\}) = +\infty \text{ if and only if } L^*(\omega, \{\omega\}) = 1.$$

A set of states $T \in B_\Omega$ is *transient* if (i) $T$ is the disjoint union of countably many *uniformly transient sets* $U_j$, that is, sets $U_j \in B_\Omega$ such that $T = \cup_j U_j$ and if (ii) for each set there is a finite constant $M_j$, such that for all states $\omega \in U_j$,

$$E_\omega[\eta_{U_j}] = \sum_{t=1}^{\infty} p^{*t}(U_j|\omega) < M_j. \tag{110}$$

The set of states $E$ is said to be $p^*$*-inessential* if

$$Q^*(\omega, E) = 0 \text{ for all } \omega \in \Omega. \tag{111}$$

Thus, a set of states, $E$, is inessential if the probability that the state process visits the set $E$ infinitely often is zero stating from any state. If a set of states is inessential, then if the process visits the set at all, it leaves the set for good after finitely many moves. The union of countable many inessential states is called an *improperly* $p^*$*-essential set*. Any other set is called *properly* $p^*$*-essential*.

Finally, the state process $\{\omega_t\}_t$ governed by $p^*(\cdot|\cdot)$ is said to be $\psi$*-irreducible* if for some probability measure $\psi(\cdot)$ on $B_\Omega$,[10]

$$\psi(E) > 0 \text{ implies } L(\omega, E) > 0 \text{ for all } \omega \in \Omega.$$

Thus if the process $\{\omega_t\}_t$ governed by $p^*(\cdot|\cdot)$ is $\psi$-irreducible, then it hits all the "important" sets of states (i.e., the sets $E \in B_\Omega$ such that $\psi(E) > 0$) with positive probability starting from any $\omega \in \Omega$. The state process $\{\omega_t\}_t$ governed by $p^*(\cdot|\cdot)$ is said to be $\psi$*-recurrent* if,

$$\psi(E) > 0 \text{ implies } Q^*(\omega, E) = 1 \text{ for all } \omega \in \Omega.$$

In addition to modeling the emergence of state-network dynamics from the feedback between strategic behavior, financial network structure, and risk, one of our main objectives is to study the stability properties of the resulting equilibrium state process as well as the implications of these stability properties for endogenous systemic risk. A key component of our analysis is the notion of a dynamic basin of attraction. Intuitively, a set of states $H \subset \Omega$ is a basin of attraction if the state process $\{\omega_t\}_t$ reaches $H$ in finite time with probability 1 and once there, stays there. The question we wish to answer is this: does the state process $\{\omega_t\}_t$ that emerges from the equilibrium interplay of strategic behavior, network structure, and risk generate basins of attraction. We begin by considering the classical notion of a Maximal Harris set of states.

## 6.4 Dynamic Basins of Attraction: Maximal Harris Sets

A set of states $H \in B_\Omega$ is called a *maximal Harris set* if there exists some probability measure $\varphi(\cdot)$ on $B_\Omega$ such that $\varphi(H) > 0$,

$$\varphi(A) > 0 \text{ implies } L^*(\omega, A) = 1 \text{ for all } \omega \in H,$$
$$\text{and}$$
$$L^*(\omega, H) = 1 \text{ implies that } \omega \in H.$$

Note that a maximal Harris set is a *maximal absorbing set* and is indecomposable. Moreover, if $H$ and $H'$ are distinct Maximal Harris sets, then they are disjoint. Finally, note that if the state process reaches a particular Harris set then it remains there for all future

[10] Here, the probability measure $\psi(\cdot)$ is a maximal irreducibility measure (see Section 4.2.2 in Meyn and Tweedie, second edition, 2009).

periods. By Proposition 9.1.1 in Meyn and Tweedie (2009), because we have $L^*(\omega, H) = 1$ for all $\omega \in H$,

$$L^*(\omega, H) = Q^*(\omega, H) = 1 \text{ for all } \omega \in \Omega.$$

Thus, if the set of states $H$ is maximal Harris, then process $\{\omega_t\}_t$ restricted to $H$ is $\varphi$-irreducible and Harris recurrent - where Harris recurrence means that $Q^*(\omega, H) = 1$ for all $\omega \in H$.

The fact that a maximal Harris set is a maximal absorbing set makes it a good candidate for a basin of attraction. But in order to fully qualify as a basin of attraction we must show that - or identify conditions under which - the state process reaches such a set in finite time with probability 1.

## 6.5 The Fundamental Conditions for Stability: Drift and Global Uniform Countable Additivity

Given the equilibrium Markov state transition $p^*(\cdot|\cdot)$ what can be said concerning stability? What conditions guarantee that the equilibrium state process, $\{\omega_t\}_t$, reaches a Harris set in finite time with probability 1. It turns out that the Tweedie Conditions (2001) do just that:

**The Tweedie Conditions** *(2001)*:
*There exists a measurable set of states $C \subseteq \Omega$, a nonnegative measurable function*

$$V(\cdot) : \Omega \longrightarrow [0, \infty],$$

*and a finite real number $b$ such that*
*(1) (the drift condition) for all $\omega \in \Omega$*

$$\int_\Omega V(\omega')dp^*(\omega'|\omega) \leq V(\omega) - 1 + bI_C(\omega),$$

*and*
*(2) (uniform countable additivity) for any sequence $\{B_n\}_n \subset B_\Omega$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),*

$$\lim_{n \longrightarrow \infty} \sup_{\omega \in C} p^*(B_n|\omega) = 0.$$

We say that the Markov transition $p^*(\cdot|\cdot)$ satisfies *global uniform countable additivity* if for any sequence $\{B_n\}_n \subset B_\Omega$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n \longrightarrow \infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0, \tag{112}$$

and we will say that the Tweedie conditions are satisfied globally if both conditions (1) and (2) hold with $C = \Omega$.

Using results due to Meyn and Tweedie (2009), Tweedie (2001), and Costa and Dufour (2005), we will show below that if the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the equilibrium state process, $\{\omega_t\}_t$, is *globally uniformly countably additive*, then the equilibrium process possesses a finite set of basins of attraction.

We have our main result on global uniform countable additivity.

**Theorem 4** *(Global Uniform Countable Additivity)*
*Suppose assumptions [A-1]-[A-5] hold. Then $p^*(\cdot|\cdot)$ is globally uniformly countably additive.*

**Proof.** : Let

$$\Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega) := \{q(\cdot|\omega, G) : (\omega, G) \in \Omega \times (\Delta(Q) \times \mathbb{G} \times \mathbb{G})\}.$$

We will show that $\Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega)$ is sequentially compact in the $\sigma(rca(\Omega), \mathcal{B}_{\Omega}^{\infty})$ topology.[11]

By the compactness of $\Omega \times (\Delta(Q) \times \mathbb{G} \times \mathbb{G})$, for any sequence

$$\{q(\cdot|\omega^n, G^n)\}_n \subset \Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega),$$

there is a subsequence, $\{q(\cdot|\omega^{n_k}, G^{n_k})\}_k$ such that $(\omega^{n_k}, G^{n_k}) \underset{\rho_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}}{\longrightarrow} (\omega^*, G^*)$ implying by assumption [A-4](7a) that for all $E \in B_{\Omega}$

$$q(E|\omega^{n_k}, G^{n_k}) \longrightarrow q(E|\omega^*, G^*) \in \Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega).$$

Thus, for each $f \in \mathcal{B}_{\Omega}^{\infty}$, we have

$$\int_{\Omega} f(\omega')q(\omega'|\omega^{n_k}, G^{n_k}) \longrightarrow \int_{\Omega} f(\omega')q(\omega'|\omega^*, G^*).$$

Thus, $\Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega)$ is sequentially compact in the $\sigma(rca(\Omega), \mathcal{B}_{\Omega}^{\infty})$ topology. By Corollary 2.2 in Lassere (1998), $p^*(\cdot|\cdot)$ is globally uniformly countably additive. In particular, letting $\{B_k\}_k \subset B_{\Omega}$ be any decreasing sequence(i.e., $B_k \downarrow \emptyset$) and $\{f_k(\cdot)\}_k$ be the sequence of functions in $\mathcal{B}_{\Omega}^{\infty}$ where for each $k$, $f_k(\omega) := I_{B_k}(\omega) \in \mathcal{B}_{\Omega}^{\infty}$, we have by Corollary 2.2 in Lassere (1998) that the sequential compactness of $\Delta_{\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})}(\Omega)$ implies that

$$\lim_{k\longrightarrow\infty} \sup_{(\omega,G)\in\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})} \int_{\Omega} f_k(\omega')q(\omega'|\omega, G) = \lim_{k\longrightarrow\infty} \sup_{(\omega,G)\in\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})} q(B_k|\omega, G) = 0.$$

Thus, because

$$\sup_{(\omega,G)\in\Omega\times(\Delta(Q)\times\mathbb{G}\times\mathbb{G})} q(B_k|\omega, G) \geq \sup_{\omega\in\Omega} q(B_k|\omega, G^*(\omega)) \geq 0,$$

we have

$$\lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} q(B_k|\omega, G^*(\omega)) = \lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} p^*(B_k|\omega) = 0.$$

■

By Theorem 2, under assumptions [A-4] the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of network and coalition formation is globally uniformly countably additive. Moreover, letting $C = \Omega$, $V(\omega) = 1$ for all $\omega \in \Omega$, and $b = 2$, the drift condition is also satisfied. Thus, under assumptions [A-4] especially (7a), we are able to conclude in Theorem 2 that the Tweedie conditions are satisfied globally (i.e., with $C = \Omega$).

# 7 Basins of Attraction, Invariance, and Ergodicity

We now have our main result concerning stochastic basins of attraction and the stability of the equilibrium state process $\{\omega_t\}_t$ governed by $p^*(\cdot|\cdot)$.

[11] $rca(\Omega)$ is the Banach space of finite signed Borel measures on $(\Omega, B_{\Omega})$ and $\mathcal{B}_{\Omega}^{\infty}$ is the Banach space of $\mu$-equivalence classes of real-valued, bounded measurable functions on $(\Omega, B_{\Omega})$.

**Theorem 5** (*Basins of Attraction: The Finite Decomposition of the State Space - a variation on Tweedie, 2001*)
*Under assumptions [A-1]-[A-5] the equilibrium state process, $\{\omega_t\}_t$, governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, G^*(\cdot))$ generates a decomposition of the state space $\Omega$ into a finite number of disjoint basins of attraction and a disjoint transient set. In particular, this decomposition is of the form*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T, \tag{113}$$

*where each $H_i$ is a basin of attraction (i.e., maximal Harris) and $T$ is transient, and has the property that for every state $\omega \in \Omega$*

$$L(\omega, \cup_i H_i) = \mu\left\{\tau_{\cup_i H_i} < \infty | \omega_0 = \omega\right\} = 1. \tag{114}$$

Because in our model the *Tweedie conditions hold globally*, it follows from Theorem 2 in Tweedie (2001) that the entire state space $\Omega$ admits a finite decomposition,

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

consisting of a finite number of indecomposable, Maximal Harris sets, $H_i$, and a transient set $T$. The key step in establishing this finite decomposition is to show that because the equilibrium Markov transition,

$$\omega \longrightarrow p^*(\cdot|\omega) := q(\cdot|\omega, G^*(\omega)),$$

is globally, uniformly countably additive (see Theorem 3 above), the state space contains at most a finite number of disjoint absorbing sets (see Tweedie 2001, Lemma 2). Moreover, by Theorem 2 in Tweedie (2001), this decomposition is such that $L^*(\omega, \cup_{i=1}^{N} H_i) = 1$ for all $\omega \in \Omega$. Thus, the state process, $\{\omega_t\}_t$, governed by the equilibrium Markov transition, $q(\cdot|\cdot, G^*(\cdot))$, is such that no matter where the process begins (no matter what state is the starting point), it reaches in finite time with probability 1 one of finitely many basins of attraction, $H_i$, and once there, stays there. Thus, our Theorem 4 is a network formation game rendition of Theorem 2 in Tweedie (2001) based upon the fact that the equilibrium Markov transition, $q(\cdot|\cdot, G^*(\cdot))$, is globally uniformly countably additive.

Our next result establishes that the equilibrium Markov transition possesses a finite number of ergodic measures, one for each basin of attraction.

**Theorem 6** (*Invariance and Ergodicity of the Process of Network and Coalition Formation - also a variation on Tweedie, 2001)*
*Suppose assumptions [A-1]-[A-5] hold. Let $\{\omega_t\}_t$ be the equilibrium state process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, G^*(\cdot))$, and let*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

*be the corresponding finite decomposition of the state space into basins of attraction. The following statements are true:*
*(1) Corresponding to each basin of attraction $H_i$, there is a unique $p^*$-invariant probability measure $\gamma_i(\cdot)$ with $\gamma_i(H_i) = 1$. Moreover, for each state $\omega = (C, D, s)$,*

$$p^{*(n)}(E|\omega) := \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L(\omega, H_i)\gamma_i(E \cap H_i), \textit{ for all } E \in B_\Omega. \tag{115}$$

*where $p^{*k}(E|\omega)$ is defined recursively, see (55).*

*(2) The set of all ergodic probability measures is given by*

$$\mathcal{E}^* = \{\gamma_i(\cdot)\}_{i=1}^N.$$

*Moreover, a probability measure $\gamma(\cdot)$ on $(\Omega, B_\Omega)$ is $p^*$-invariant, i.e. $\gamma(\cdot) \in \mathcal{I}^*$, if and only if $\gamma(\cdot)$ is given by*

$$\gamma(E) = \sum_{i}^{N} \gamma(H_i)\gamma_i(E \cap H_i), \text{ for all } E \in B_\Omega. \tag{116}$$

*(3) $\mathcal{E}^*$ is a singleton (i.e., $\mathcal{E}^* = \{\gamma(\cdot)\}$) if and only if the state process $\{\omega_t\}_t$ is $\psi$-irreducible, in which case for each state $\omega = (C, D, s)$ and for every set of states $E \in B_\Omega$*

$$\frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \gamma(E).$$

**Proof.** (1) Under our assumptions [A-1]-[A-5] (see the proof of Theorem 4 above), $p^*(\cdot|\cdot)$ satisfies the Tweedie conditions globally. As a result, the first statement in part (1) is an immediate consequence of Lemma 5 in Tweedie (2001). The second statement also follows from the fact that in our model the Tweedie conditions hold globally and Theorem 1 in Tweedie (2001) (also, see Chapter 13 in Meyn and Tweedie 2009).

(2) Again because the Tweedie Conditions are satisfied globally, the first statement in part (2) follows from Lemma 2 in Tweedie (2001), Theorem 2.18 part (1) in Costa and Dufour (2005), Theorem 3.8 in Costa and Dufour, and the proof of Proposition 5.3 in Costa and Dufour. The second statement in part (2), that $\gamma(\cdot) \in \mathcal{I}^*$ implies (116), follows from the proof of Proposition 5.3 in Costa and Dufour (2005). The fact that (116) implies $\gamma(\cdot) \in \mathcal{I}^*$ follows from observation (but also, see Theorem 19.25 in Aliprantis and Border 2006 and Theorem 2 in Villareal 2004).

(3) Finally, because the Tweedie Conditions are satisfied globally, necessary and sufficient conditions for $\mathcal{E}^*$ to be a singleton, given in terms of $\psi$-irreducibility follow from Theorem 3 in Tweedie (2001). The convergence result in part (3) follows from the convergence result in part (1) of the Theorem and the fact that if there is only one basin of attraction $H$ (i.e., one maximal Harris set), then by Theorem 3, $L^*(\omega, H) = 1$ for all $\omega \in \Omega$. ■

Note that the probability measures in $\mathcal{E}^*$ are *orthogonal* (i.e., the ergodic measures live only on the basins of attraction) that is, for all $i$ and $i'$ in $\{1, 2, \ldots, N\}$ with $i \neq i'$,

$$\gamma_i(\Omega \backslash H_i) = \gamma_{i'}(H_i) = 0.$$

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