INCONSPICUOUSNESS AND OBFUSCATION: HOW LARGE SHAREHOLDERS DYNAMICALLY MANIPULATE OUTPUT AND INFORMATION FOR TRADING PURPOSES

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Abstract. The large shareholder literature examines how a large shareholder trades off the advantage of being able to influence the boardroom decisions of the firm, while small shareholders free ride on the outcomes, against the extra risk entailed in large shareholdings. A large shareholder can also profit because his ability to affect fundamentals improves his ability to hide his private information from other informed traders and from market makers in his stock market dealings. In a static version of the model, the large shareholder increases the volatility of firm fundamentals, but by adjusting his trading strategy this increase is of the component of his private information that is unforecastable by the market maker: he obfuscates. As a result, market liquidity falls.

I then use Fourier transform methods, including a new spectral factorisation algorithm, to construct a continuous time dynamic version of the large shareholder model. In the dynamic model, the large shareholder does not just simply amplify the fundamental value of the firm as in the static model: he also alters the fundamental autoregressive structure of the fundamental value process because this improves his ability to hide his private information from other informed traders and from market makers, that is, to obfuscate. As a consequence, the real fundamental processes of the firm are induced to resemble noise trade structurally. The model thus marries market microstructure outcomes with real resource allocation.

1. Introduction

Globalstar is a satellite communications company that has a fleet of 48 satellites in low earth orbit. The satellites are operationally similar to moving cell phone towers: they field signals from ground-based phones and other devices and redirect the signals to ground stations where they are forwarded to other phones or to the internet. The company's niche is the absence of lags in the signals because the satellites are close to the ground, in contrast with companies that operate satellites in geostationary orbit.

The company's enterprise value is approximately three billion dollars as of this writing, and it is publicly traded, but it is 57 per cent owned by

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insiders. The dominant one by far is Jay Monroe; Monroe's ownership flows mostly through a hedge fund, Thermo Capital, that he owns and controls. Monroe is both the CEO and the chairman of the board of Globalstar.

Monroe is privy to information about the firm that outside shareholders do not have, and is able to act on that information by altering decisions about products, production, personnel and myriad other matters—and can keep these decisions from the view of outside shareholders. Should Globalstar, which is in the main a satellite communications company, enter a new market, say to sell spectrum that it owns for satellite communication, that would compete against other companies that are in the private network business? Monroe would have inside knowledge, and could approve that entry or not—and also trade on that information.

Monroe does in fact trade shares in Globastar. (Because he is officially an insider, he is required to disclose these trades after two business days, but they are carried out confidentially at the time of execution.[1] [2]) Monroe's trades, and trading-related activity such as debt financing and acquisition negotiations that are initially carried on in secret, affect the market price of Globalstar. Knowing that he is a large shareholder, it is incumbent on Monroe to temper not only his trading but also his business decisions within Globalstar.

The perspective of the existing literature is that by being a large shareholder in Globalstar, Monroe faces risks that can be potentially deleterious

[1]This is similar in intent to the requirement for large shareholders to report their trades within the 60 days leading up to filing their acquisition of a five per cent or greater fraction of a company's shares, with filing required within ten days of the acquisition. See [14].

[2]Monroe is also officially enjoined from trading on "material information," that is, inside information about the firm. But it is equally important to note that Monroe's activities to alter the fundamentals of the firm are not prohibited, nor are back door trades such as option execution, the acquisition or discharge of debt, the size of dividends, or decisions on the issuance of new shares. Indeed the large shareholder literature by its very character presumes that large shareholders are privy to private inside information, otherwise the large shareholder could not benefit from his actions, and small shareholders would have no grounds for free riding on his actions—but the literature (such as DeMarzo and Uroševic [18]) does not explicitly model information in the large shareholder's problem.

An exception is Collin-Dufresne and Fos [14]: they have activist shareholder paying a cost $C(\omega)$ where ω is effort to affect the terminal value of the firm. Mechanically, this effort increases the mean payoff directly rather than the variance. In addition, the market only has a noisy signal of the activist's initial holding X_0. One could thus view this as "material information." In any case, the insider is in fact allowed to trade on his information, and his effort comes only *after* he has accumulated his position at T.

The entire Kyle model and also the Glosten-Milgrom literature presumes that the insider does in fact have material information, but never really discusses the channel through which it is acquired. Similarly, the DeMarzo and Uroševic [18] paper does not really grapple with the material information issue.

When one considers that there are about 4,000 US publicly traded firms, each of which has many employees with material information, versus about 4,200 employees in the SEC only a few of whom are attorneys and who deal with myriad issues besides insider trading, the conclusion that insider trading in some dimension is common seems ineluctable.

to him,[3] but his ability to monitor the firm and make production decisions can enable him to appropriately minimize those risks. Smaller shareholders might benefit from his incentives and receive lower returns in exchange for the monitoring incentive—that is, they might free ride: key papers include Admati, Pfleiderer and Zechner [1], Shleifer and Vishny [32], and De Marzo and Uroševic [18].

Monroe's possession of private information about the underlying fundamentals of the firm suggests an alternative view: that he will attempt to use that information in trading. The thrust of this paper is that the combination of large shareholder status and Monroe's interest in trading on his private information will lead him to actually alter the fundamentals of the firm, and in a way that rests fundamentally on a deep property of the Kyle model.

In pure trading models where informed large traders cannot affect firm value but do have private information as with the descendants of the standard Kyle [28] model such as Back [3], Back, Cao and Willard [5], Holden and Subrahmanyam [25], and Foster and Viswanathan [21], and also Kyle's 1989 paper [29], information is used by the market makers and by the informed rivals from current or past prices to impute the information of the informed traders. The informed traders know this, and attempt to trade on the part of their private signals that is unforecastable using public price information. As a result, total order flow looks like noise trade order flow; the informed traders hide behind the noise traders. The informed trades are thus *inconspicuous*. This inconspicuousness appears in other studies: Danilova [15], who coined the term, and Back and Baruch [4], each using very different technical frameworks, find this result.

As in other versions of the Kyle model, the expected profit of the informed trader is proportional to the product of the volatility of the fundamental value and the volatility of the noise trade. If the informed trader is also a large shareholder, he can affect the fundamental value of the stock by his actions, and so he can affect that profit by increasing the volatility of the value. That strategy emerges here.

What is key is that the amplification is not merely the enlargement of the variance of the fundamental. In a static version of the model, the large shareholder increases the volatility of firm fundamentals in this way, but because he simultaneously adjusts his trading strategy, from the market maker's perspective the increase in volatility is of the component of his private information that cannot be forecasted by market makers—the inconspicuous part of his trades. Thus, he obfuscates.

In reality, the shocks that impinge on firm value are dynamic. Inconspicuousness still emerges as a strategy: privately informed traders alter the dynamic (i.e. autoregressive) structure of their trades so that total order

[3]Assuming that due to contractual or practical strictures he cannot diversify and balance his portfolio appropriately.

flow has the same dynamic structure as noise trade. But in addition, the large shareholder can affect the underlying fundamental value of the firm not only in the conventional static sense, he can affect the dynamic structure of the value, that is, its autoregressive structure. He can profit by altering this structure, because this improves his ability to hide his private information from other informed traders and from market makers. The large shareholder now not only increases volatility, he alters the serial correlation structure of the firm's fundamental value process.

Altering the fundamentals distorts in the allocation of resources in the real activities of firms. Thus, the view that stock markets operate to allocate capital to firms is far too simplistic: markets fundamentally alter firms.

Outline. That the large shareholder alters the time series structure of the fundamental is the central result of the paper. The result is highlighted in two ways: first, with a proof that pure amplification of the underlying fundamentals is suboptimal, and second, with a numerical characterization of how the time series structure is actually altered.

To lay the groundwork for these results I first briefly review the literature on the volatility impact of managerial and shareholder incentives on firm value volatility. I then set out a static model with the ingredients of the basic Kyle (1985) [28] model: stocks, informed traders, market makers and noise traders, but with the additional feature that the the informed trader is also a large shareholder who can affect the fundamental value of the first stock via a costly action.

By adding the additional feature that the informed traders base their trades not just on their private signal but also on the price, the model becomes equivalent to the Kyle (1989) [29] setting. This is because with an assumed Gaussian distribution of the fundamental value of the firm and also Gaussian noise traders, the inherent quadratic objectives of the informed traders and the market makers leads to linear demand curves. Because they are linear one can completely solve for them in close form by finding their intercept and slope coefficients.

This equivalence carries over to the the dynamic model. Now the intercept and slope coefficients apply to histories of private signals and histories of prices, and so they effectively become functions that can be found using fixed point methods. Moreover, the fixed points of these functions can be interpreted in terms of time series properties such as persistence that in turn have economic meaning; more standard approaches aren't able to generate these interpretations.

To find these functions I use linear operator and control theory, known to economists as frequency domain methods, which allow a succinct characterization of how endogenous dynamics are affected by incentives and by equilibrium considerations; most of the details about the methods are in a technical appendix, including a new algorithm for the key step of spectral factorization. I present a couple of basic results regarding the trading behavior of the informed traders, namely that in equilibrium they hide their trades so as to be inconspicuous—a result that is fairly trivial to show in the frequency domain setting—and I note how this relates to other modeling approaches. I then add the large shareholder to the dynamic model and demonstrate with these methods that his actions will alter the dynamic structure—that is, the time series or autoregressive structure—of the firm's fundamentals.

The solution strategy of the paper can be summarized in the following table.

Model	Problem	Direct	$\rightarrow$	Indirect
Static	Informed trader	Choose market order X conditional on private signal and on price, i.e. choose demand as in Kyle (1989)		Linear demand schedule choice: Choose intercept and slope coefficient
	Market maker	MM chooses linear pricing rule to equal conditional forecast of value		MM chooses λ to minimize forecast error variance
Dynamic	Informed trader	Choose market order X(t) at each t conditional on private signal history and price history		Frequency domain: Choose linear filters on history of private signals and price, conditional on price filter Λ to determine $X(t)$ process
	Market maker	MM chooses linear pricing rule to equal conditional forecast of value		MM chooses Λ filter on order flow history to minimize norm of forecast error variance

The standard approaches are summarized in the "direct" column; the methods that are used here are summarized in the "indirect" column. The models will be set up and described in the direct fashion, but the solution is facilitated, especially in the dynamic model, using the indirect method. The equivalence of these approaches has been thoroughly explored and established in previous papers such as [33], [9] and [6].

2. Literature

The conclusion that the large shareholder will amplify the volatility of that part of the firm's fundamentals that he observes privately seems intuitively reasonable: if you give a CEO options in the company stock, his incentive is to increase the volatility of the company stock price in order to increase the option price.[4,5] This has already been noted by the literature: recent references include Camara and Henderson [12], Goldman and Slezak [22], Peng and Roell [31], Goldstein and Guembel [23] and Bolton, Scheinkman and Xiong [10].

Peng and Roell empirically document the increase in shareholder litigation when managers are given option contracts as incentives. Their conclusion is that such contracts encourage the managers to focus on short term share prices.

Bolton, Scheinkman and Xiong develop a model in which the short term focus of the managers when they are given such option contracts is potentially desirable, because it enables them to increase the short term speculative component of the share price, benefiting current shareholders. That finding is mirrored in a sense here because the model here includes noise traders who are the source of profit for the informed traders. Both Peng and Roell and Bolton, Scheinkman and Xiong focus on managerial behavior, rather than large shareholder behavior.

Goldman and Slezak develop a model in which managers can exert agency-style effort to manipulate earnings so as to increase the value of their incentive pay. They conclude however that the manager's increased effort can actually increase shareholder welfare because the efforts are in the right direction, that is, they improve firm value.

Goldstein and Guembel develop a model of price manipulation that is more focused on the production side. Firm managers observe prices in the market and make investment decisions based on those prices because they contain information that the manager might not be able to directly observe. Informed speculators, distinct from the managers, know this and manipulate prices to profit from the manager's response, rather than directly to their own private signals. This notion is reflected in the model here: the large shareholder observes market prices (including the prices of other stocks that might have correlated information) and acts on that information as well as his own.

Camara and Henderson analyze the effects of several types of incentive contracts, and the effects of penalties and risk aversion on manager behavior. Among other conclusions, they find that risk aversion limits the manager's incentive to increase firm volatility.

There is also a literature, exemplified most recently by Edmans and Manso [20] in which large shareholders, referred to as blockholders, trade in markets that are modeled in standard microstructure fashion, in order to impose discipline on managers. Specifically, managers hold stock in the firm, and firm value is positively affected by their efforts; blockholder trades reveal

[4]Options can be a substantial part of pay: when Meg Whitman took over Hewlett-Packard, she was given compensation package consisting of a salary of $1 and options.

[5]Pay in the form of direct equity holdings also has the same properties as options due to the limited liability aspect of equity, and this effect is of course magnified in leveraged firms. I thank Christian Julliard for this observation.

information about firm value and thus indirectly reward that effort. These models are static in nature however, and so the feedback from trades into actions that influence firm value are only indirect.

Finally, a recent paper by Collin-Dufresne and Fos, [14] has a model that is close in some respects to the model in this paper: an activist shareholder pays a cost $C(\omega)$ where ω is effort to affect the terminal value of the firm. (Mechanically, this effort increases the mean payoff directly rather than the variance.) In addition, the market only has a noisy signal of the activist's initial holding of the stock, so there is a private information element. The insider trades on his information, but, unlike this paper, his effort comes only after he has accumulated his position at the terminal time.

3. The static model

The static model is similar to the model of Bernhardt and Taub [8], which in turn builds on the demand-schedule model of Kyle [29]. Firm value without action by the large shareholder is

$$v = e \tag{1}$$

where e is Gaussian $N(0, \sigma_e^2)$. An informed trader[6] receives a zero-mean Gaussian-distributed signal e of the value of the firm whose stock is traded.[7] The informed trader can affect the value of the firm via his actions, that is, the informed trader is also the large shareholder.

As in the standard Kyle (1985) model [28], there is a competitive market maker who sets price conditional on total order flow, which consists of the informed trader's order and noise trade. The informed trader submits orders based not only on his private information but also based on this price. As shown in Bernhardt and Taub [6], this construct, because of the quadratic objectives and Gaussian shocks, is equivalent to the Kyle (1989) model [29] in which the strategy space consists of demand schedules. Because those demand schedules are linear, this is in turn equivalent to a problem in which the informed trader optimizes over the linear coefficients of the demand schedules.

To analyze the model I will first set out the optimization problems of the informed traders and then show how it can be transformed to an optimization problem in demand schedules; the large shareholder's optimization of the firm fundamentals can be treated separately as a consequence of the transformation. I then state several results about the optimization of the firm fundamentals.

[6]In the literature on the Kyle model, the informed traders can also be called insiders or speculators.

[7]It is straightforward to consider multiple stocks as was done in [8], but I will assume a single stock here.

The large shareholder. The large shareholder can be viewed as dividing his efforts between trading activities that benefit only himself and public activities that he is required to carry out for the benefit of the firm. To enhance his trading profits, the large shareholder chooses θ to weight his private signal in order to alter fundamental value:

$$v = e - \theta e. \tag{2}$$

Because the fundamental e is constructed to have a zero mean, it is immediately evident that the influence of θ will ultimately be on the *variance* of the fundamental e.

The large shareholder's other activity is to act for the benefit of the firm, that is, as a fiduciary. These actions can be monitored and rewarded only on the publicly observable elements of firm fundamentals. The public fundamental does not enter in the valuation of the company for trading purposes precisely because it is public.[8] There is therefore a separability between the public and private parts of the firm.

To capture the fiduciary aspect of the large shareholder within the linear-quadratic structure of the Kyle model, the large shareholder's actions on behalf of the company might then be expressed as exerting effort to hit a publicly observable target driven by a random variable η, independent of the privately observable fundamental e. The incentive for the large shareholder to hit the target can be represented by a penalty function

$$-C(\eta - \theta\eta)^2.$$

with the same θ as the coefficient of the private fundamental in (2). The large shareholder thus faces a tradeoff: his actions to alter the private fundamental via his choice of the amplification factor θ will thus detract from his efforts to hit the public target.

Because this penalty function does not directly interact with the pricing part of the model, and because only the variance of η affects the optimization, it is more tractable to express it as a simple penalty function on θ, with the influence of the fundamental e alone:

$$-C(\theta e)^2.$$

This simplification avoids complications that would otherwise camouflage the main result.

Trades. As is standard there is uninformed noise trade u. The informed trader's trade x (whether or not he is the large shareholder) is a linear function of his private signal of value e, expressed as trading intensity coefficient b, and of the net information in price, with intensity γ; total order flow is the sum of informed and uninformed trades $x + u$.

[8] That is, because they are public, market makers price elements of the firm that are driven by public shocks perfectly, and so informed traders cannot profit by trading on that information.

Pricing. Market makers receive the aggregate orders from the informed trader and from the noise traders, but are unable to distinguish individual trades. Market makers are competitive, so they construct price to be efficient, that is, to be an optimal forecast of the value v.

I focus only on linear equilibria in which the informed trader's trade is linear; because the noise trade and fundamental value are Gaussian, the optimal forecast is a linear projection. Price is thus a linear function of order flow

$$p = \lambda\,(x + u)$$

where λ is a projection coefficient that expresses the signal extraction that is being carried out. The solution of linear projection is equivalent to finding the pricing coefficient λ that minimizes the forecast error variance. [9]

The large shareholder's problem. The large shareholder's objective is a weighted sum of his trading profits, as enhanced by his amplification of the fundamental e via his choice of θ, and the minimization of the penalty from failing to hit the public target, also via θ.

Because θ operates directly on the fundamental e, this problem is potentially ill-posed: the optimal θ would then be a function of the noise trade realization as well as the fundamental e. To avoid the ill-posedness problem I assume that the choice of θ can be conditioned on e, but not on the direct observation of the noise trade. The maximization then occurs in two stages:

$$\max_{\theta} E\left[\max_{x}\left[\left(e - \theta e - \lambda\,(x+u)\right)x - \frac{C}{2}(\theta e)^2 \middle| (1-\theta)e, p\right] \middle| e\right] \tag{3}$$

where C is a constant penalty on the magnitude of the large shareholder's realized amplification θe. In the first stage, the large shareholder chooses the optimal trading strategy x conditional on the modified fundamental, $(1-\theta)e$, and potentially on the noise trade, imputed via the price, if there is only one informed trader. In the second *ex ante* stage he chooses the amplification factor θ.

The indirect approach to the large shareholder's optimization problem. To solve the optimization problem in equation (3), it is helpful to transform the problem as outlined in the introduction. The first step is to convert the informed trader's trade conditioned on price to his trade conditioning on the information in price.

Lemma 1. *Let the informed trader's order flow take the linear form*

$$x = b(1-\theta)e + \gamma(b(1-\theta)e + u) \tag{4}$$

[9]In fact, linearity of pricing is not immediate, but it can be shown that a linear equilibrium exists, and this paper focuses only on this possibility. The linearity of the pricing rule is developed in Back [3]. Further analysis of the uniqueness of linear equilibria is presented in Boulatov, Kyle and Livdan [11], and also Bernhardt and Taub [9].

Then the net information in price is

$$b(1-\theta)e+u \tag{5}$$

Proof: Applying the linear pricing rule to total order flow yields the price

$$\lambda(x+u)=\lambda(b(1-\theta)e+\gamma(b(1-\theta)e+u)+u)=\lambda(1+\gamma)(b(1-\theta)e+u) \tag{6}$$

Dividing by $\lambda(1+\gamma)$ yields the net information in price. □

Correspondingly, because of the quadratic structure of the problem, it is straightforward to show that, with the assumption of linear pricing, the optimal x in equation (3) is a linear function of the relevant information sources, equation (4). Given the equivalence established in Lemma 1, equation (4) is then essentially a demand schedule in the spirit of Kyle's (1989) model [29]. I will refer to the coefficients of that demand schedule, b and γ, as *trading intensities.*

Holding θ fixed, then because the optimal x is a linear function of the fundamentals, it is possible to demonstrate that the problem of maximizing over x is equivalent to maximizing over the trading intensities b and γ after taking the *unconditional* expectation with the assumption that x is linear; this equivalence is demonstrated for example in Bernhardt and Taub [8], and is briefly outlined in Appendix A. This equivalent problem is more convenient analytically because θ can then be chosen simultaneously with b and γ. This is the approach I will use going forward, and will be especially key when solving the dynamic model.[10]

Formalizing this:

Definition 2. *The large shareholder's strategy space consists of a triple* $(b,\gamma,\theta)\in R^3$.

Expressing the trading strategies explicitly in terms of b and γ as in equation (4), then carrying through the expectation in this way and with the assumption that the private signals are uncorrelated, the optimization problem for the large shareholder is

$$\max_{\{b,\gamma,\theta\}}\left\{((1-\theta)-b\,(1+\gamma)\,\lambda)\,(1+\gamma)b\sigma_e^2-(1+\gamma)\,\lambda\gamma\sigma_u^2-\frac{C}{2}\theta^2\sigma_e^2\right\} \tag{7}$$

The two-stage structure is no longer needed.

A definition of equilibrium can now be stated:

Definition 3. *A linear Bayesian Nash equilibrium is a tuple* $(b,\gamma,\theta,\lambda)\in R^4$ *such that*

[10] The outlines of the equivalence can be seen by noting that the coefficients b and γ in the informed trader's optimal order, using the "direct" approach as expressed in equation (4), will be functions of the linear pricing coefficient λ. Because λ is a projection coefficient, it is a function of the variances of the fundamental and of the noise trade, σ_e^2 and σ_u^2, and so in equilibrium b and γ will be functions of these variances as well. In the indirect approach the unconditional expectation renders the variances as coefficients in the objective, and optimizing over b and γ then leads more directly to the solutions expressed in terms of the variances. See Appendix A for a more direct demonstration.

(i) *A trading and amplification strategy consisting of a triple* $(b, \gamma, \theta) \in R^3$ *solves* (7)*, conditional on the linear pricing rule* λ*, with information set* (e, u)*, and*

(ii) *The linear pricing rule* λ *minimizes the market maker's forecast error variance of* $E\left[v \middle| b(1-\theta)e + \gamma(b(1-\theta)e+u)+u\right]$ *conditional on the informed trader's strategy* (b, γ, θ) *and the information from total order flow* $b(1-\theta)e + \gamma(b(1-\theta)e+u)+u$.

3.1. **The large shareholder's first order conditions.** The first-order condition for b can be written as follows:

$$[((1-\theta) - b(1+\gamma)\lambda)(1+\gamma) - b(1+\gamma)\lambda(1+\gamma)]\,\sigma_e^2 = 0$$

with solution

$$b = \frac{1}{2\lambda(1+\gamma)}(1-\theta). \tag{8}$$

Thus, the large shareholder's trading intensity on private information is modified by his alteration of the variance of the fundamental.

The first order condition for γ is

$$\begin{aligned}((1-\theta) - b(1+\gamma)\lambda)\, b\sigma^2 \\ - (1+\gamma)\lambda\sigma_u^2 - (b\lambda)(1+\gamma)b\sigma_e^2 - \lambda\gamma\sigma_u^2 = 0\end{aligned} \tag{9}$$

After substituting the solution for the b and the other γ this reduces to

$$\frac{1}{2}(1-\theta)\, b\sigma^2 - (1+\gamma)\lambda\sigma_u^2 - (b\lambda)(1+\gamma)b\sigma_e^2 - \lambda\gamma\sigma_u^2 = 0$$

A further solution for γ requires the formula for the optimal pricing rule λ which will be derived below.

The first-order condition for θ is

$$-(1+\gamma)b\sigma_1^2 - C\theta\sigma_e^2 = 0.$$

with solution

$$\theta = -\frac{(1+\gamma)b}{C}. \tag{10}$$

The coefficient γ has the following interpretation: it is the (negative of the) projection coefficient of the informed trader's trade on public information onto publicly available information, namely price; in turn, price is informationally equivalent to total order flow.[11] Because γ is negative (see the solution below), the factor $1+\gamma$ is the coefficient of the forecast error of that projection. Thus, θ is proportional to the market maker's forecast error coefficient $(1+\gamma)$ on the traded part of the large shareholder's private signal, b; this is the same quantity on which the informed trader's orders are based.

[11] These assertions are elaborated in Bernhardt, Seiler and Taub [9] and in Seiler and Taub [33].

3.2. **Equilibrium.** To solve the model the solution of [6] can be directly applied, with a single informed trader. To apply the model with the large shareholder effect, I amend the notation to reflect the large shareholder's modification of the fundamental value of the firm. From the market maker's perspective that fundamental value is $(1-\theta)e$ with variance $(1-\theta)^2\sigma_1^2$. Define the modified fundamental and fundamental variance,

$$\tilde{e} \equiv (1-\theta)e \qquad \tilde{\sigma}_e \equiv (1-\theta)\sigma_e \tag{11}$$

Using the formulas for the equilibrium quantities on page 7 of [6] (changing to the notation of this paper, so that b is the coefficient on the private signal) the solutions are:

$$b = \frac{1}{2\lambda(1+\gamma)} \quad \gamma = -\frac{b^2\tilde{\sigma}_e^2}{b^2\sigma_e^2+\sigma_u^2} \quad \lambda = \frac{1}{(1+\gamma)}\frac{b\tilde{\sigma}_e^2}{b^2\tilde{\sigma}_e^2+\sigma_u^2} \tag{12}$$

where λ is the market maker's linear least squares projection on order flow, and with the second equality following because $N=1$. Solving the three equations yields the equilibrium quantities in terms of fundamentals:

$$b = \frac{\sigma_u}{\tilde{\sigma}_e} \quad \gamma = -\frac{1}{2} \quad \lambda = \frac{\tilde{\sigma}_e}{\sigma_u} \tag{13}$$

3.2.1. *The effect of the large shareholder.* The solution for θ can now be characterized.

Proposition 4. *θ is negative, resulting in amplification of the variance of the fundamental.*

Proof: The full solution for θ is complicated because $\tilde{\sigma}$ is itself a function of θ. Substituting from equation (13) in equation (10),

$$\theta = -\frac{\frac{\sigma_u}{(1-\theta)\sigma_e}}{2C} \tag{14}$$

yielding a quadratic in θ, with one negative solution and one positive solution. For large values of C, these solutions approach 1 and 0 respectively; clearly the positive solution, which would reduce the variance of the fundamental and incur a large penalty, is suboptimal. The proof that the negative solution is optimal is as follows. The reduced form for profit in terms of θ can be found by substituting the solutions for b and θ into the reduced form solution for profit in equation (13), but with the volatility altered to account for the large shareholder's action, as in equation (15):

$$\pi = \frac{\tilde{\sigma}_e\sigma_u}{2} = \frac{(1-\theta)\sigma_e\sigma_u}{2}$$

There is also a penalty term in the objective for the large shareholder. The positive solution of the quadratic equation is larger in absolute value than the negative solution, and therefore the penalty term is larger in magnitude (more negative) than the negative solution. Therefore the negative solution is optimal. $\square$

We can now establish that an equilibrium exists using these solutions and also characterize it.

Corollary 5. *A linear Bayesian Nash equilibrium exists.*

Proof: From equation (14) in Proposition 4, the solution for θ is determined by the exogenous elements. The solution for θ then determines the effective variance of the fundamental, $\tilde{\sigma}_e$. The fundamentals $\tilde{\sigma}_e$ and σ_u then determine the equilibrium b, γ and λ using results from Bernhardt and Taub [6]. □

Because θ is negative, the modified fundamental $e - \theta e$ is in fact an amplification of the fundamental variance. But in addition, note that this term too is proportional to the forecast error coefficient $1 + \gamma$. Thus, the amplification is only on the unforecastable part of the fundamental, limited only by the penalty C. I summarize the result as follows.

Proposition 6. *The large shareholder amplifies the unforecastable part of his component of the fundamental.*

Proof: Using a projection algebra argument, Bernhardt and Taub ([6], p. 11) demonstrate that the order flow x is comprised of a linear function of the market maker's forecast error of the informed trader i's private signal e. For the large shareholder, this translates to

$$x = \frac{1+\gamma}{\lambda} E\left[(1-\theta)e + \left| \left((1-\theta)e - E\left[(1-\theta)e \middle| x+u\right]\right)\right.\right]$$

where it should be noted that the inner expectation is conditioned on total order flow $x + u$. Factoring $1 - \theta$ out of this expression, we have

$$x = (1-\theta)\frac{1+\gamma}{\tilde{\lambda}} E\left[(1-\theta)e + \left| \left(e - E\left[e \middle| x+u\right]\right)\right.\right] \tag{15}$$

where $\tilde{\lambda}$ is the pricing coefficient when there is a large shareholder. By Proposition 4, θ is negative, and so the key effect of the large shareholder is to not only amplify the private signal, but to amplify the *unforecastable part* of the signal. □

Corollary 7. *The large shareholder's amplification of the unforecastable part of his component of the fundamental increases his profit.*

Proof: Using formula (12) from [6], profit is

$$\pi = \lambda(1+\gamma)\sigma_u^2 = \frac{\tilde{\sigma}_e \sigma_u}{2}. \tag{16}$$

Because $1 - \theta$ exceeds unity, the large shareholder's action serves to *amplify* the fundamental value e and thus his profit. □

Thus, the large shareholder does in fact conform to intuition: he will effectively increase the volatility of the firm's value, and he takes advantage of this excess volatility in the dimension in which he receives signals in his trading. Thus, the large shareholder has a larger profit than a correspondingly informed outsider; these profits come at the expense of the noise traders.

The increase in the effective fundamental variance increases the pricing coefficient λ, as would be expected: the signal to noise ratio has been increased. The volatility of price also increases:

$$E\left[(\lambda(1+\gamma)(b(1-\theta)e+u))^2\right] = \frac{\tilde{\sigma}_e^2}{2}$$

In addition, the forecast error variance also increases. This is not obvious a priori, because the higher variance of the fundamental σ_e^2 raises the signal to noise ratio in order flow. This would be expected to increase λ—which it does—and thus reduce the overall forecast error variance. However, the way the signal is amplified is via the forecast error of the signal, and intuitively this should not improve the signal component, and this is the result.

Proposition 8. *The large shareholder amplifies the market maker's forecast error variance.*

Proof: The forecast error variance is

$$E\left[((1-\theta)e-\lambda(x+u))^2\right] = \frac{\tilde{\sigma}_e^2}{2} \tag{17}$$

By proposition 4 θ is negative, so the forecast error variance exceeds the forecast error variance when there is no large shareholder. □

The fact that the large shareholder amplifies the *non-forecastable* part of his order flow—obfuscation—suggests that in a dynamic setting the large shareholder might want to alter the time series structure of the fundamental. This conjecture is true and in the next section I set the groundwork for demonstrating this. The method used in the static model—restricting actions to be linear functions of the information realizations, and then solving for the coefficients of those linear polices—works in the dynamic setting as well, but requires functional analysis tools.

4. Adding dynamics

I will now set out a dynamic version of the model using a continuous time approach. I use the continuous-time analogue of Bernhardt, Seiler and Taub [9] and Seiler and Taub [33] in order to carry out the dynamic analysis. The main tools are the Laplace and Fourier transforms and the continuous-time analogue of the Wiener-Hopf equation. These tools are described in sections 6.A (pp. 216-220), 7.1-7.2 (pp. 221-228), and 7.A (262-264) of Kailath, Sayed and Hassibi [27]. An additional reference is Hansen and Sargent [24].

The dynamic model differs significantly from the more standard approach exemplified by the paper of Back [3], which uses a PDE approach to solving the dynamic Kyle model. In the original dynamic Kyle model the fundamental value of the firm is fixed, as is the time horizon. The only dynamic fundamental element of the model is the noise trade, which is Brownian motion. Because of the fixed horizon, the behavior of the equilibrium is strongly influenced by boundary conditions. In the model of this paper, the firm's fundamental is itself a dynamic process, as is the signal observed by the informed trader. While increasing the complexity of the model in many dimensions, it also renders it stationary. The stationary model can be mapped to the frequency domain and then solved with essentially algebraic methods. More concretely, the transformed model reduces the solution process to finding functions that are the analogues of the coefficients b, γ, θ and λ from the static model, namely $B(\cdot)$, $\Theta(\cdot)$, $\Gamma(\cdot)$ and $\Lambda(\cdot)$, which are elements of a Hilbert space. The optimization problems of the informed traders and of the market makers can be expressed as variational problems in the frequency domain with optimization over these functions. Because the functions are in essence the generating functions of stochastic processes, they have clear economic interpretations, particularly with regard to the persistence of the processes.

In the standard market microstructure literature, the noise trade is assumed to be a Brownian process so that incremental noise trades are serially uncorrelated, while there is a single realization of the underlying asset value, as in the original Kyle model [28], or another Brownian process as in Danilova's recent model [15], so that information are highly persistent. By contrast, in this paper while I maintain the assumption of a persistent value process, I allow the degree of persistence to be a variable. This serves to highlight the starkly different dynamic structure of order flow and prices, differences that highlight the economic forces driving those processes.

In the next section I develop the technical elements of the dynamic model without the large shareholder, building on previous work. The conclusions regarding inconspicuousness are easily established in this framework. In subsequent sections I add the large shareholder and develop the main result: that the large shareholder obfuscates not only by amplifying the non-forecastable part of informed order flow, but by altering the fundamental of the firm itself.

4.1. **Technical preliminaries.** I begin with the assumption that the total cost of shares $X(t)$ held by a trader at each time t is proportional to the discounted cost of acquiring them at each instant, $\int_0^t e^{-rs}p(s)x(s)ds$, where $x(s)ds$ is the incremental shares acquired, r is the discount rate and $p(t)$ is the price of a share at time t.

In the standard setup of the Kyle model, the underlying value is fixed, and is revealed at the terminal time T. There are two differences here: first, the horizon is infinite, and so the underlying value is effectively never revealed. Second, the value fluctuates stochastically, and meaning must be given to this fluctuation. The interpretation will be that at each moment there is a possibility that the firm will be bought, merged or terminate, with some probability that is independent of past or current states and associated hazard rate δ; I will refer to this as *conversion*. Should the conversion occur, the discounted payoff for an informed trader is

$$e^{-rt}\overline{v}(t)\int_0^t x(s)ds$$

where $\overline{v}(t)$ is the time-t realization of the stochastically evolving fundamental value, and this happens with probability $\delta e^{-\delta t}$. The probability-weighted expected payoff at time t is then

$$E\left[e^{-rt}\delta e^{-\delta t}v(t)\int_0^t x(s)ds\middle|\omega(0)\right]$$

where $\omega(t)$ is the trader's information at t. Thus, the expected profit over all dates of conversion is

$$E\left[\int_0^\infty e^{-rt}\delta e^{-\delta t}\overline{v}(t)\int_0^t x(s)\,ds\,dt\middle|\omega(0)\right]$$

By changing the order of integration we can write the inner terms as

$$\int_0^\infty x(t)\int_t^\infty e^{-rs}\delta e^{-\delta s}\overline{v}(s)\,ds\,dt \tag{18}$$

Defining the probability-discounted value of the asset at any moment as

$$v(t)\equiv e^{-\delta t}\int_t^\infty \delta e^{-(\delta+r)(s-t)}\overline{v}(s)\,ds$$

we can write equation (18) as

$$E\left[\int_0^\infty e^{-rt}x(t)v(t)dt\middle|\omega(0)\right]. \tag{19}$$

This then justifies writing discounted expected profit as

$$E\left[\int_0^\infty e^{-rt}(v(t)-p(t))x(t)dt\middle|\omega(0)\right]. \tag{20}$$

Value and noise trade process specifics. Let $e(t)$ be the private information process for the informed trader, who is also the large shareholder. I assume that the process is a Gaussian, zero-mean white noise process, and it remains the case that the firm value is equal to this signal. Now however, the signal and consequently the firm value are stochastic processes. The raw or unmodified firm value process is a filtered version of this fundamental signal process:[12]

$$v(t)=\int_0^\infty \phi(\tau)e(t-\tau)d\tau \tag{21}$$

It deserves emphasis that it is the probability-discounted payoff, $v(t)$, that is defined using these primitives, rather than the "raw" payoff $\overline{v}(t)$.

[12]See Appendix B for a discussion of the interpretation of white noise in continuous time.

It is possible to rewrite (21) as a stochastic integral, namely

$$v(t) = \int_0^\infty \phi(\tau) dZ(t-\tau) \tag{22}$$

where $Z(t)$ is a Brownian motion, using Doob's [19] and Hansen and Sargent's [24] stochastic integral notation. If the large shareholder modifies the fundamental, then he would choose an additional filter $\theta(\cdot)$ to convolve with the v process:

$$\int_0^\infty \theta(\tau) dv(t-\tau) \tag{23}$$

so that the modified fundamental would be

$$\tilde{v}(t) = \int_0^\infty \phi(\tau) dZ(t-\tau) - \int_{\sigma=0}^\infty \int_0^\infty \theta(\sigma)\phi(\tau) dZ(t-\tau-\sigma) \tag{24}$$

Similarly, the noise trade process can be a filtered version of a fundamental white noise process $n(t)$:

$$u(t) = \int_0^\infty \nu(\tau) n(t-\tau) d\tau.$$

but it is mathematically more proper to formulate the process as a convolution of Brownian increments, that is,

$$u(t) = \int_0^\infty \nu(\tau) dN(t-\tau). \tag{25}$$

where $N(t)$ is an independent Brownian process.

Both the value and noise processes are characterized by the filters ϕ and ν. For the purposes of characterizing the model it will be assumed when necessary that the value and noise trade processes are Ornstein-Uhlenbeck processes, that is, analogues of autoregressive processes in discrete time settings. The filters are then in exponential form:

$$\phi(\tau) = e^{-\rho\tau} \qquad \nu(\tau) = e^{-\eta\tau}.$$

In the limit, these processes become Brownian motions at $\rho = 0$ and $\eta = 0$, and white noise processes at the limit of the other extreme, $\rho = \infty$ and $\eta = \infty$. Economic intuition suggests that the value processes should be highly predictable; similarly intuition suggests that noise trade should not be persistent. To keep the model tractable I will assume that noise trade is white noise ($\eta = \infty$), but that the value process can have any degree of persistence, characterized by $0 \leq \rho < \infty$.

Using the continuous time transform as described in Kailath, Sayed and Hassibi [27] p. 217, and also in Appendix C, the s-transforms of the filters for the value and noise trade processes are then $\Phi = \frac{1}{s+\rho}$, which is the transform of an Ornstein-Uhlenbeck process (see Davis [17] p. 80), and the identity matrix I respectively (again, see Davis [17] p. 80).

4.2. **Order flow and public information.** Before carrying out the full transformation of the objective to the frequency domain, it will be helpful to convert the expression of the large shareholder's trade from demand submission form as in Lemma 1, in which the large shareholder reacts directly to price, to an equivalent one in which the large shareholder bases his trade on the public information inherent in price. To keep the argument simple I will temporarily drop θ, the large shareholder's modification of the fundamental from the notation: the large shareholder will be treated as if he were a conventional informed trader, and I also temporarily drop the filter on the fundamental process, ϕ.

Let $\Omega(t)$ be the public information process, which is going to be equivalent to the information in price.[13] In keeping with the assumption of *linear* strategies as in the static model, the informed trader's trading strategy process is restricted to be a linear filtering of the histories of these processes:

$$x(t) = \int_{\tau=0}^{\infty} (b_\omega(\tau) dZ(t-\tau) + b_\Omega(\tau)\Omega(t-\tau)d\tau).$$

Moreover, the linear filters $b_\omega(\cdot)$ and $b_\Omega(\cdot)$ that constitute the trading strategy are assumed to be fixed, that is, they are independent of time. As with the static model, the filters, including that for the large shareholder, are assumed to operate directly on the fundamental processes; the impact of the large shareholder is felt via the value process and its impact on price.

The trading strategy filters chosen by the informed trader or large shareholder are elements of the space of square-integrable functions, taking account of discounting:

$$L_2(r) \equiv \left\{ f(\cdot) : \Big| \int_0^\infty e^{-rt} \left| f(t) \right|^2 < \infty \right\} \tag{26}$$

However, the focus will not be on the objective with this choice set, but rather elements of the transformed objective and control filters, which I will detail later.

In addition, noise traders exogenously submit an order flow process $u(t)$. Adding up the informed and noise trade yields total order flow:

$$x(t) + u(t) = \left(\int_{\tau=0}^{\infty} (b_\omega(\tau) dZ(t-\tau) + b_\Omega(\tau)\Omega(t-\tau)d\tau) \right) + u(t). \tag{27}$$

Defining the left hand side of (27) as $\Omega(t)$, we have the recursion

$$\Omega(t) = \int_{\tau=0}^{\infty} b_\omega(\tau) dZ(t-\tau) + \int_{\tau=0}^{\infty} b_\Omega(\tau)\Omega(t-\tau)d\tau + u(t). \tag{28}$$

$\Omega(t)$ is a stochastic process that also defines the market maker's information process. It will be key for the further analysis to express this information process in modified form.

[13]In the sequel it will be understood that the information is expressed as a filtration $\mathcal{F}_t$ that is adapted to $\Omega(t)$, but because the transform methods that will later facilitate the solution will operate on $\Omega(t)$, this will be the focus from now on.

The right hand side of the $\Omega(t)$ process in equation (28) is a convolution as described in Kailath, Sayed and Hassibi [27] p. 217, but the integration is one-sided. However, if it is known in advance that the functions b_ω and b_Ω are one-sided, the integral can be converted to two-sided form and obtain the Laplace transform of the equation.[14]

Following Kailath, Sayed and Hassibi's convention of using capital letters for the Laplace transform, the transform of equation (28) is:

$$O(s) = B(s)e(s) + B_\Omega(s)O(s) + U(s).$$

where $B(s)$ is the transform of b_ω (see pp. 216-217 of [27]). The convolutions have been converted into products as a result of the transform. With this transformation, it is now possible to solve for $O(s)$ with straightforward algebra. This yields

$$O(s) = (1 - B_\Omega(s))^{-1}(B(s)e(s) + U(s)) \tag{29}$$

The solution approach used in Bernhardt et al [9] can now be followed: define γ as the filter characterized by the transform

$$\Gamma(s) = (1 - B_\Omega(s))^{-1} B_\Omega(s) \tag{30}$$

and then substituting from equation (28) into the order flow equation (27) and using (29) and (30), the order flow process becomes

$$\begin{aligned} x(t) = &\int_{\tau=0}^{\infty} b_\omega(\tau)dZ(t-\tau) \\ &+ \int_{\tau=0}^{\infty}\gamma(\tau)\left(\int_{\sigma=0}^{\infty} b_\omega(\sigma)dZ(t-\tau-\sigma) + u(t-\tau)\right)d\tau \end{aligned} \tag{31}$$

The bracketed term can then be viewed as the public information process inherent in the price process.

Maintaining the assumption of linear pricing, the price process is determined by a linear filter λ, with transform Λ, applied to total order flow:[15]

$$\begin{aligned} p(t) = \int_0^\infty \lambda(\tau)\Bigg[&\int_{\sigma=0}^{\infty}\int_{\nu=0}^{\infty} b_\omega(\sigma)(1+\gamma(\nu))\,dZ(t-\nu-\sigma-\tau)d\sigma \\ &+ \int_{\nu=0}^{\infty}\gamma(\nu)u(t-\nu-\tau)d\nu\Bigg]d\tau \end{aligned} \tag{32}$$

These ingredients will now be combined to form the objective for the informed traders.

---

[14] See a further elaboration of the technical details of this transform in Appendix B.

[15] Again, it should be noted that the linearity of the price process here is an assumption; as previously noted the validity of this assumption for existence has been explored in previous papers such as [9]; the necessity of linear pricing in the standard Kyle model was established in [3].

4.3. **The informed trader's objective.** Resuming the explicit incorporation of $\theta(\cdot)$ and $\phi(\cdot)$, and expressing the informed trader's actions in terms of the filters expressing the value process in equation (21), the price process in (32), and the order flow process from (31) which includes the public information process, we can write the time-domain objective (20) for the large-shareholder informed trader as

(33)
$$
\begin{aligned}
&\max_{\{b_\omega(\cdot),\gamma(\cdot),\theta(\cdot)\}} E\int_0^\infty e^{-rt}\Bigg(\int_{\tau=0}^\infty \phi(\tau)dZ(t-\tau)-\int_{\tau=0}^\infty\int_{\sigma=0}^\infty \phi(\tau)\theta(\sigma)dZ(t-\tau-\sigma)\\
&-\int_{\tau=0}^\infty \lambda(\tau)\Bigg[\int_{\sigma=0}^\infty\int_{\nu=0}^\infty b_\omega(\sigma)\left(1+\gamma(\nu)\right)dZ(t-\nu-\sigma-\tau)d\sigma\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\int_{\nu=0}^\infty \gamma(\nu)u(t-\nu-\tau)d\nu\Bigg]d\tau\Bigg)\\
&\times\left(\int_{\tau=0}^\infty b_\omega(\tau)dZ(t-\tau)+\int_{\tau=0}^\infty \gamma(\tau)\left(\int_{\sigma=0}^\infty b_\omega(\sigma)dZ(t-\tau-\sigma)+u(t-\tau)\right)d\tau\right)dt
\end{aligned}
$$

where $b_\omega(\cdot)$ and $\gamma(\cdot)$ are elements of the space of exponentially bounded functions that are analytic and square integrable, that is, they are in $L_+^2(r)$ in Hansen and Sargent's notation ([24]), with the integration adjusted for discounting, and with the optimization over the filters directly rather than over informed order flow x. With these elements in place, it is possible to state a preliminary definition of equilibrium:

Definition 9. *An linear equilibrium is a quadruple* $(b_\omega(\cdot),\gamma(\cdot),\theta(\cdot),\lambda(\cdot))$ *elements of* $L_2(r)$ *such that*

(i) *The functions* $(b_\omega(\cdot),\gamma(\cdot),\theta(\cdot))$ *solve the informed trader/large shareholder's optimization problem* (33) *taking as given* $\lambda(\cdot)$ *with information* $((e(t-s),u(t-s))_{s=0}^\infty$ *for all* t*, and*
(ii) *The filter* $\lambda(\cdot)$ *in the market maker's pricing rule in* (32) *is the conditional forecast of the value process at every time* t*, taking as given the informed trader's trading rules* $(b_\omega(\cdot),\gamma(\cdot),\theta(\cdot))$.

The objective in (33) is nontrivial, because the choice of the optimal action each period is conditioned on information, which includes the history of endogenous actions, and similarly the pricing rule in (32) is a fairly complicated object. As in the static model in which the solution could be approached by first taking the expectation of the fundamentals and then seeking solutions of the optimal coefficients, the solution of the dynamic model is more straightforward if the objective is first converted to frequency domain form, with the choice variables converted from time-domain period-by-period actions to the choice of optimal Laplace transformed *filters* in the frequency domain.[16] I next develop the recipe for converting the objective to frequency domain form.

At this point that the assumption that the choice objects b_ω, γ and so on, are restricted to be linear filters, means that the equilibrium, if it exists, will be a linear equilibrium. While the uniqueness of the equilibria in the Kyle model has long been a subject of exploration in the literature, I am imposing this linear structure as an assumption.

Once the conversion to the frequency domain occurs, the optimal filters are found by solving a *static* variational problem. It deserves emphasis that in the original time-domain statement of the informed trader's objective and the market maker's filtering problem, it is at least conceptually possible for the optimal filters b_ω, γ and so on to be nonstationary functions of time, that is at each time t the informed trader would choose filters $b_\omega(t,\cdot) \neq b_\omega(s,\cdot)$ for $s \neq t$. However, because the problem is time-separable with linear constraints and a quadratic objective, this will not be the case: the optimal filters will be stationary. This in turn means that the frequency domain version of the model in which the optimal filters are chosen directly also solves the time-domain version of the model.

5. Optimizing in the frequency domain

Whiteman [36] constructed a discrete time model and then converted the objective itself into z-transform form. The optimization was then over linear operators or filters that were found via a variational derivative of the transformed objective.[17] This was achieved by imposing the constraint that the controls must be a linear filter of the information, and taking the expectation of the objective prior to optimizing over those filters; this is the extension of the similar operation that was carried out in going from equation (3) to equation (7) in the static model. However, it is essential to reduce the covariance function of the fundamental processes—the white noise fundamentals—to a scalar covariance matrix. In continuous time, the equivalent operation is to make the fundamental covariance function $R_x(t)$ a Dirac δ-function.[18]

If the fundamental processes are serially uncorrelated, as is the case here by the assumption that the fundamental processes are white noise, then the expectation of an objective like (33) leaves an integral in which the integrand consists of products of functions. Fourier transforming these objects then yields a convolution in the frequency domain, and the variational derivative of these convolutions can then be calculated. Proceeding in this way with abstract functions f and g,

$$\int_0^\infty e^{-rt} f(t) g(t) dt = \int_{a-i\infty}^{a+i\infty} F(s) G^*(r - s^*) ds \tag{34}$$

[16] The equivalence of these formulations was explored in [9].

[17] An earlier instance of the method is in Davenport and Root [16].

[18] Again, see Davis [17], pp. 80-81.

where the notation G^* signifies the complex conjugate transpose of G, $G^*(r - s^*)$.[19] The $r - s^*$ term captures discounting, and where the integration is along a strip parallel to the imaginary axis in which $\mathrm{Re}(s) = a$, where the functions F and G are analytic in the right half plane—that is, F and G have no poles or singularities in the region $\mathrm{Re}(s) > -r$, and with a small enough to avoid poles and thus yield convergence, that is, $a < r$.[20] There are two parts to the integrand: the "causal" part $F(s)$ and the "anti-causal part" $G^*(r - s^*)$, reflecting the inner product that is expressed in the objective.

The right hand side of (34) defines an inner product; formally, define

Definition 10. *H^2 is the set of square integrable functions on the right half plane with inner product defined in* (34) *such that*

$$H^2(r) = \left\{ F : \int_{a-i\infty}^{a+i\infty} F(s)F^*(r - s^*)ds < \infty \right\}.$$

I will seek equilibria consisting of functions from this space.

5.1. **The large shareholder's problem.** In the static model, the market maker views the modified fundamental $(1-\theta)e$ as indistinguishable from a "raw" fundamental; similarly, the large shareholder solved for the optimal signal intensity coefficient b and the price information intensity coefficient γ by treating the modified fundamental as a raw fundamental; the choice of the amplification coefficient could then be developed subsequently. This same strategy works in the dynamic model.

The large shareholder chooses the filter $\theta(\cdot)$ on his private signal in order to alter fundamental value:

$$v_t = \int_0^\infty \phi(\tau)dZ(t-\tau) - \int_0^\infty \int_0^\infty \theta(\sigma)\phi(\tau)dZ(t-\tau-\sigma)d\sigma$$

There are no intrinsic restrictions on the filter, other than that it must by analytic, i.e., it can be backward looking but not forward looking. Thus, the filter can be designed so that the stochastic structure in the underlying shocks is altered to be more or less persistent, and to have additional structure such as zeroes that make the price process noninvertible in some appropriate sense.

The penalty on the large shareholder's action is

$$E\left[\left(C^{1/2}\int_0^\infty \int_0^\infty \theta(\sigma)\phi(\tau)dZ(t-\tau-\sigma)d\sigma\right)^2\right], \tag{35}$$

that is, the amplification is treated as a penalty process. The motivation for this penalty is the same is in the static model: one can think of the large shareholder as having a responsibility to hit stochastic targets that are

[19] For a concrete example in which this integration is calculated, see the proof of Lemma 15 in Appendix D.

[20] Notice that as in the discrete time case discounting weakens the constraints on poles.

public, as well as being interested in altering the privately observed fundamentals for the purpose of making trading profits. As in the static model, the essential features of the target-hitting formulation can be captured by a penalty formulation.[21]

Applying the transform method to the informed trader's problem, the transformed objective will be a function of the value process filter $\Phi(s)$, the public information process filter $O(s)$, the modification of the fundamental $\Theta(s)$, the pricing filter $\Lambda(s)$, and the informed trader's order flow $X(s)$. The transform of the objective (33) is then

$$\max_{X,\Theta} \int_{a-i\infty}^{a+i\infty} \operatorname{tr}\Big(\Phi(s) - \Theta(s)\Phi(s)) - \Lambda(s)O(s))X^*(r-s^*)R \\ - \frac{C}{2}\Theta(s)\Phi(s)\Phi^*(r-s^*)\Theta^*(r-s^*)\Big)ds \tag{36}$$

where O is the transform of the total order flow process from equation (29), and where $\Phi(s)$ is the (vector) value process Laplace transform, the valuation process reflects the action of the large shareholder, with $\Theta(\cdot)$ the transform of $\theta(\cdot)$. Because I solve the model using the indirect approach—that is, taking expectations on the assumption of linearity and then solving for the optimal filters—it is appropriate to simultaneously solve for X and Θ.

The internal pieces of X and Φ, Λ, and O can now be broken out. The causal and anti-causal pieces ($\Phi - \Lambda O$ and X^* respectively) are such that the Fourier transform of a sum is the sum of the Fourier transforms. The convolution of functions of the s operator translates into multiplication of functions in the s-domain: let $g(t) = \int_0^\infty h(\tau)m(t-\tau)d\tau$, and consider the Fourier transform of $\int_0^\infty f(\sigma)g(t-\sigma)d\sigma$. Then it is immediate that

$$F(s)G(s) = F(s)(H(s)M(s)). \tag{37}$$

Thus, the double integration in equation (35)) is an iterated convolution, and thus the transform results in an iterated product $\Theta(s)\Phi(s)$; see the discussion surrounding equation (37).

With this result in hand one can write the Fourier transformed objective (36) with the explicit decomposition of the price process. Also, $B(s)$ is the Fourier transform of the filter $b_\omega(t)$; $\Gamma(s)$ is the Fourier transform of $\gamma(t)$, and H is the Fourier transform of $1+\gamma(t)$, so that

$$H(s) = 1 + \Gamma(s). \tag{38}$$

With these ingredients the large shareholder-informed trader's transformed order flow filters are a vector of transforms

$$\begin{pmatrix} B(1+\Gamma) & \Gamma \end{pmatrix} \tag{39}$$

[21] In fact this is potentially a bit more complicated. In the dynamic setting, the filter associated with the public target process is added to the forecast error filter in the first order condition for θ. That in turn potentially complicates the solution of θ. The complication has been avoided here by in essence assuming that the public target is a constant, zero, yielding the penalty function used in the main text.

with each element corresponding to the filters operating on the fundamental process $e(t)$ and on the noise trade fundamental $u(t)$ via the public information process; the extra term in the first element adds the informed trader's direct operation on his own signal. Similarly, the price process transform consists of the elements

$$\Lambda \begin{pmatrix} BH & H \end{pmatrix}$$

operating on the individual fundamental process $e(t)$ and the noise trade process $u(t)$, and where the total order flow by all agents is captured by adding up the individual transforms in (39) and using the compact notation in (38). Finally, Φ is the Fourier transform of the raw unmodified firm value process process that the informed trader/large shareholder sees in his signal, and which is the value process of the stock.

Combining these ingredients yields the s-transform of the objective (33). By assuming independence of the noise trade and fundamentals, we can write the objective in detail as

$$\max_{\{B,\Gamma,\Theta\}} - \int_{a-i\infty}^{a+i\infty} \operatorname{tr} \left\{ \begin{pmatrix} (\Phi - \Theta\Phi) - BH\Lambda \\ -H\Lambda \end{pmatrix} \begin{pmatrix} (1+\Gamma^*)B^* & \Gamma^* \end{pmatrix} R - \frac{C}{2}\Theta\Phi\Phi^*\Theta^* \right\} ds \tag{40}$$

where the causal and anti-causal parts reflect the inner product that is expressed in the objective, and where R is the covariance matrix function of the Dirac-δ fundamentals $e(t)$ and $u(t)$.[22] To keep the model tractable, I will assume as in [9] and [33] that the noise trade process is uncorrelated with the fundamental value and signal processes, so the covariance function R is block diagonal:

$$R = \begin{pmatrix} R_e & 0 \\ 0 & R_u \end{pmatrix}. \tag{41}$$

which parallels equation (7) of [9].

It is key that the optimization in equation (40) is now over the *functions* B, Γ, and Θ. Before solving this problem, I state the similarly transformed problem of the market maker.

5.2. **The market-maker's objective.** Similarly, the market-maker's objective can be stated in the frequency domain. The market-maker strives to minimize the forecast error variance of price conditional on order flow:

$$\max_{\{\Lambda\}} - \int_{a-i\infty}^{a+i\infty} \operatorname{tr} \left\{ \begin{pmatrix} (\Phi - \Theta\Phi) - BH\Lambda \\ -H\Lambda \end{pmatrix} \begin{pmatrix} (\Phi^* - \Phi^*\Theta^*)^* - \Lambda^* H^* B^* & -\Lambda^* H^* \end{pmatrix} R \right\} ds \tag{42}$$

Again, it is key that the optimization in this problem is over the *function* Λ.

[22] Again, see [27] p. 218 or [24] p. 209.

5.3. **Equilibrium in the frequency domain.** Having translated the model to the frequency domain, it is now possible to restate the equilibrium definition for the frequency domain version of the model. In the original time domain version of the problem, the equilibrium is a set of stochastic processes. Those processes can be characterized by the filters on the fundamental processes, and those filters when transformed are functions in H^2 that can be treated as algebraic objects that are the solutions of simultaneous equations. Solving for those functions implicitly solves for the equilibrium time-domain functions via inverse transforms.[23]

Definition 11. *A stationary dynamic linear Bayesian Nash equilibrium is a trading strategy triple* $(B(\cdot), \Gamma(\cdot), \Theta(\cdot))$ *of elements in* H^2 *and a linear pricing rule* $\Lambda(\cdot) \in H^2$ *such that*

(i) *The trading strategy solves the informed trader-large shareholder maximisation problem* (40)*, conditional on the linear pricing rule and the information set characterized by R;*
(ii) *The linear filter* $\Lambda(\cdot)$ *minimizes the market maker's forecast error variance in equation* (42) *with the information in order flow, conditional on the informed trader's trading strategy* $(B(\cdot), \Gamma(\cdot), \Theta(\cdot))$

5.4. **Solving the transformed model.** As in the static model, the large shareholder can take as given the modified fundamental process, as expressed by the term $(1-\Theta)\Phi$, as the exogenous fundamental process when calculating the optimal trading strategy functions B and Γ, and then calculate the optimal Θ separately. In keeping with the notation in the earlier analysis I will denote the value process after manipulation by the large shareholder by $\tilde{\Phi}$. The large shareholder acts on his private signal which has filter Φ, so the modified value process filter might be

$$\tilde{\Phi} = \big(\Phi(s) - \Theta(s)\Phi(s)\big)'$$

5.5. **The solution for** B**.** Following the steps in [9], the first-order conditions of the s-transformed objectives can be stated. First, the notation

$$\mathcal{A}^*$$

denotes an arbitrary function is the s-domain that is anti-causal, that is, $A^*(s) = 0$, for s in the right half-plane.

The variational first-order condition for B in the large shareholder's objective (40) is

$$\Big[\Big(\tilde{\Phi} - BH\Lambda\Big)(1+\Gamma^*) - B(1+\Gamma)\Lambda^* H^*\Big]\sigma_e^2 = \mathcal{A}^*,$$

which is a Wiener-Hopf equation. In the uncorrelated case the elements are all scalars and will commute; the solution methods for continuous-time

[23]Because the frequency domain solutions for the functions have clear economic interpretations, I will not actually carry out the inverse transform.

Wiener-Hopf equations outlined in Kailath, Sayed and Hassibi section 7.A can now be used. Gather terms to restate the equation as

$$B\Big[\Lambda H(1+\Gamma^*)+(1+\Gamma)H^*\Lambda^*\Big]\sigma_e^2=\tilde{\Phi}(1+\Gamma^*)\sigma_e^2+\mathcal{A}^* \tag{43}$$

Now propose a factorization

$$GG^*\equiv\Lambda H(1+\Gamma^*)+(1+\Gamma)H^*\Lambda^* \tag{44}$$

where by standard results G can be chosen to be analytic and invertible. Then the solution is

$$B=\Big\{\tilde{\Phi}(1+\Gamma^*)G^{*-1}\Big\}_+G^{-1} \tag{45}$$

where the projection operator $\{\cdot\}_+$ is defined by

$$\{F(s)\}_+=0,\qquad \mathrm{Re}(s)\le 0$$

Some interpretation of (45) is possible. The solution for B is the s-transform analogue of a projection coefficient. There are two elements in the "numerator" or covariance part of this projection coefficient: $\tilde{\Phi}$, the filter characterizing the informed trader's information, and $1+\Gamma$. As in the static setting, Γ is itself (the negative of) a generalized projection coefficient of the informed trader's order flow filter on his private signal against the total order flow.

The "denominator" of (45) is the analogue of the variance of that part of the price process that is driven by this forecast error. The solution for B in (45) is therefore the forecast error of the projection of the informed trader's information against the net information in total order flow.

Before developing the first-order condition for Γ the first-order condition for the market maker will be developed. That condition will be applied to simplify the large shareholder's problem.

5.6. **The solution for the pricing filter Λ.** The market maker's first order condition is

$$\begin{pmatrix}-H^*B^* & -H^*\end{pmatrix}R\begin{pmatrix}\tilde{\Phi}-BH\Lambda\\ -H\Lambda\end{pmatrix}=\mathcal{A}^* \tag{46}$$

where the matrices have been transposed under the trace operator. Also, the two separate terms of the first-order condition have been consolidated into a single one by taking the conjugate-transpose of the second term. Defining the function J via the factorization

$$J^*J\equiv\begin{pmatrix}B^* & 1\end{pmatrix}R\begin{pmatrix}B\\ 1\end{pmatrix}=\begin{pmatrix}B^* & I\end{pmatrix}R\begin{pmatrix}B\\ I\end{pmatrix}=B^*R_eB+R_u. \tag{47}$$

It is then possible to write the first-order condition as

$$H^*J^*JH\Lambda=H^*\begin{pmatrix}B^* & I\end{pmatrix}R\begin{pmatrix}\tilde{\Phi}\\ 0\end{pmatrix}+\mathcal{A}^*=B^*R_e\tilde{\Phi}+\mathcal{A}^*.$$

where in the last step the block-diagonal structure of R has been used. Note also that the filter characterizing the total order flow process is JH.

Multiplying both sides by H^{*-1},

$$J^* J H \Lambda = B^{*\prime} R_e \tilde{\Phi} + \mathcal{A}^* \tag{48}$$

with solution

$$\Lambda = H^{-1} J^{-1} \left\{ J^{*-1} B^{*\prime} R \begin{pmatrix} \tilde{\Phi} \\ 0 \end{pmatrix} \right\}_+ \tag{49}$$

The interpretation of the solution in (49) is straightforward. The total order flow process process is implicitly defined by the filter J. The solution for Λ is then the s-transform analogue of the projection coefficient of the true value process on total order flow.

I next return to the first-order condition for Γ, making use of the market-maker's first-order condition (46).

The solution for Γ. The variational first-order condition for Γ is

$$\begin{pmatrix} B^* & 1 \end{pmatrix} R \begin{pmatrix} \tilde{\Phi} - BH\Lambda \\ -H\Lambda \end{pmatrix} + \begin{pmatrix} -\Lambda^* B^* & -\Lambda^* \end{pmatrix} R \begin{pmatrix} B(1+\Gamma) \\ \Gamma \end{pmatrix} = \mathcal{A}^* \tag{50}$$

Substituting from the market-maker's first-order condition (46), the first term drops out, yielding

$$\begin{pmatrix} -\Lambda^* B^* & -\Lambda^* \end{pmatrix} R \begin{pmatrix} B(1+\Gamma) \\ \Gamma \end{pmatrix} = \mathcal{A}^*$$

Eliminating the Λ^* term yields

$$\begin{pmatrix} B^* & I \end{pmatrix} R \begin{pmatrix} B(1+\Gamma) \\ \Gamma \end{pmatrix} = \mathcal{A}^*.$$

Using the block-diagonal structure of R yields

$$J^* J \Gamma = -B^* R_e \begin{pmatrix} B \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{51}$$

with solution

$$\Gamma = -J^{-1} \left\{ J^{*-1} B^{*\prime} R_e \begin{pmatrix} B \\ 0 \end{pmatrix} \right\}_+ \tag{52}$$

As was pointed out above, the solution (52) is the s-transform analogue of (the negative of) the projection coefficient of the informed trader's filter on his private information against the information in total order flow.

This fact can be used to interpret the informed trader's order flow strategy. Examining the informed trader's frequency domain objective in (40), the informed trader's order flow process is characterized by the vector of the filters

$$\begin{pmatrix} B(1+\Gamma) & \Gamma \end{pmatrix}$$

acting on the vector of processes $\begin{pmatrix} e(t) & u(t) \end{pmatrix}'$; the Γ terms express the projection on the information in price. Interpreting Γ as negative—as was the case in the static example—this projection is subtracted from direct trade process on the private information itself, that is from the filter b acting directly on the process $e(t)$.[24] The interpretation is that the informed trader knows that any information the market makers can infer about his private information will be incorporated in price and thereby its profit potential neutralized. The informed trader thus trades only on the residual, unforecastable part of his private signal.

The transform method lends itself well to establishing this "inconspicuousness" result, and the associated result that the price process must be structurally equivalent to the value process itself, because the processes of the model can be characterized by their poles and zeroes:

Proposition 12. *In an equilibrium,*

(i) *The large shareholder executes his trades so that total order flow has the same autoregressive structure as the noise trade, that is, he hides and is inconspicuous;*
(ii) *The autoregressive structure of the price process is identical to the autoregressive structure of the fundamental process.*

Proof: This is a corollary of Propositions 16 and 17 in Appendix E. □

The solution for Θ. I now turn to the first order condition for Θ, the large shareholder's amplification factor. A key assumption now comes into play: that the large shareholder ignores the influence of the amplification factor Θ on the pricing filter Λ, which appears in equation (49); in this sense the Bayesian Nash assumption is a binding constraint. The resulting variational first order condition is

$$-\Phi^* \left(B(1+\Gamma)\right) R_e - (\Theta\Phi\Phi^* + \Phi\Phi^*\Theta^*)\frac{1}{2}C = \mathcal{A}^* \tag{53}$$

Consolidating via the conjugate transpose yields

$$-\Phi^* \left(B(1+\Gamma)\right) R_e - \Theta\Phi(s)\Phi^* C = \mathcal{A}^* \tag{54}$$

Dividing out Φ^* then yields

$$-\left(B(1+\Gamma)\right) R_e - \Theta\Phi(s) C = \mathcal{A}^*$$

The solution of the first-order condition for Θ (54) is then straightforward:

$$\Theta = -C^{-1}B(1+\Gamma)R\Phi^{-1} \tag{55}$$

Substituting this into the objective, the Φ^{-1} term cancels the Φ term in the objective, leaving

$$\tilde{\Phi} = \Phi + B(1+\Gamma)C^{-1}. \tag{56}$$

Recall that Γ is the transform of $\gamma(\cdot)$, which in turn is the dynamic analogue of the static projection coefficient γ. The expression $I+\Gamma$ is the transform of the forecast error filter. Thus, as with the static model, the term that

[24] The reader is reminded that these operations take place under integrals: see Appendix B.

is added on to the raw fundamental process Φ by the large shareholder is simply the unforecastable part of the large shareholder's trade!

6. Optimal obfuscation

In this section I carry out three tasks: First, I provide the main result of the paper: that the optimal Θ necessarily alters the fundamental stochastic structure of the firm. Second, I outline how existence might be demonstrated. Third, I develop a numerical example to illustrate the main result about the structure of Θ.

Demonstrating the non-constancy of Θ. The large shareholder does not just amplify the unforecastable part of his trades, he also alters the time series structure of the fundamental process itself in order to improve trading profits. This is expressed as the non-constancy of Θ. To demonstrate this I make the following assumption:

Assumption 13. *The unmodified fundamental process Φ has only one pole.*

The assumption allows the direct application of some useful theorems about the algebra of functions in the frequency domain, particularly Lemma 15 in Appendix D.

Proposition 14. *Let Assumption 13 be met. Then the optimal Θ is not a constant and therefore $(1-\Theta)\Phi$ is not proportional to Φ, that is, the large shareholder alters the autoregressive structure of the firm's fundamentals.*

Proof: The proof is by contradiction. Examining equation (55), $\Theta = -C^{-1}B(1+\Gamma)R\Phi^{-1}$, the result will follow if it can be demonstrated that $(1+\Gamma)B$ is not of order Φ. Recalling the solution for B from equation (45), $B = \left\{\tilde{\Phi}(1+\Gamma^*)G^{*-1}\right\}_+ G^{-1}$, we have

$$G^{-1}\left\{G^{*-1}\tilde{\Phi}H^*\right\}_+ H$$

Applying the annihilator lemma, Lemma 15, under the hypothesis that Θ is a constant and therefore that $\tilde{\Phi}$ is proportional to Φ, we have

$$\sim G^{-1}H\Phi$$

To establish that this expression not of order Φ, we can equivalently demonstrate that $\sim G^{-1}H$ is not a constant. Recalling the definition of G from equation (44),

$$|G|^2 = GG^* \equiv [\Lambda HH^* + HH^*\Lambda^*] = (\Lambda+\Lambda^*)HH^* \tag{57}$$

Defining

$$\Lambda+\Lambda^* \equiv |L|^2$$

By Proposition 17 in Appendix E, Λ is of order Φ; therefore $L(s)$ is a function that has nontrivial zeroes as well as the same poles as Φ. Thus, $L(s)$ is the product $N(s)\Phi(s)$, and $N(s)$ is a non-constant function. we can then write

$$G = LH$$

and so $G^{-1}H = L^{-1}$ is a non-constant function, and so $G^{-1}H\Phi$ is proportional to $N(s)^{-1}$, which is not proportional to Φ. $\square$

Thus, the large shareholder *dynamically* obfuscates: he not only amplifies the fundamental value process, he alters its time series structure by the market makers' forecast error. In the numerical computation it is possible to say more: in attempting to amplify the non-forecastable parts of the fundamental, the large shareholder attempts to mimic the noise trade process.

6.1. **Existence and characteristics of equilibrium.** A proof of the of existence of equilibrium would use the iterative approach of [33] and is beyond the scope of this paper. The complexity of the problem can be seen by considering how the equilibrium would work if Proposition 14 were not true, that is, if Θ were in fact a constant. Conditional on a solution for the constant Θ, the modified fundamental process $(1-\Theta)\Phi$ is determined, and an equilibrium can be generated using the recursion approach of [33] to determine B, Γ and Λ. The optimal constant Θ could then be determined from the solution for Θ in equation 55. However, as developed in Proposition 14, the solution will have the property that Λ, the pricing filter, will be a function of Θ even though this dependence is ignored by the large shareholder when optimizing due to the Nash assumption. Thus, to find the *equilibrium* Θ it would be necessary to solve a fixed point problem for the *system* of equations for B, Γ, Λ and also Θ,

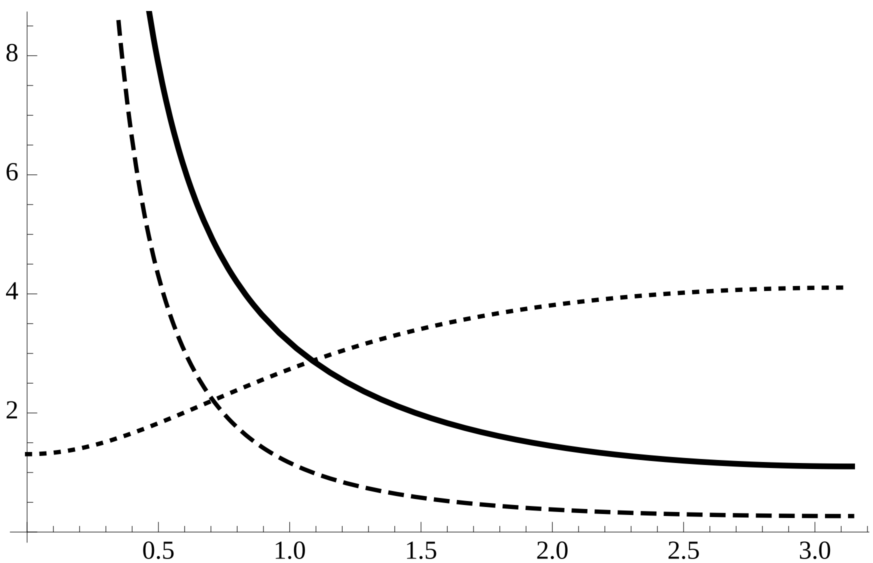
$$\begin{aligned}
B &= \left\{\tilde{\Phi}(1+\Gamma^*)G^{*-1}\right\}_+ G^{-1} \\
\Gamma &= -J^{-1}\left\{J^{*-1}B^{*\prime}R_e\begin{pmatrix} B \\ 0 \end{pmatrix}\right\}_+ \\
\Lambda &= H^{-1}J^{-1}\left\{J^{*-1}B^{*\prime}R\begin{pmatrix} \tilde{\Phi} \\ 0 \end{pmatrix}\right\}_+ \\
\Theta &= -C^{-1}B(1+\Gamma)R\Phi^{-1}
\end{aligned}$$

taking account of this dependency. Whilst a recursion leading to a fixed point for the system with a fixed Θ can be developed along the lines of the similar problem in Seiler and Taub [33], the addition of the Θ equation adds significant complexity.

Numerical characterization. As a step in the direction of at least characterizing the nature of the equilibrium, I next carry out a numerical calculation that would amount to a first step in an iterative approach: I posit that the dependence of Λ on Θ right hand side of the solution for Θ, equation (55), is ignored, and calculate the optimal iterated Θ on the left hand side in order to roughly characterize the large shareholder's strategy.

Stacking the first order conditions for B and Θ from equations (43) and (53) (see equation (64) in Appendix F) yields a matrix system with a coefficient matrix that can be factored to determine the solutions for B and Θ and then the entire system. Carrying out the factorization and plotting the spectral density of the Φ and $(1-\Theta)\Phi$ filters shows that Θ amplifies high frequencies more than low frequencies—that is, it actually reduces the persistence of the fundamental (see Figure 1). The intuition for the emphasis on high frequencies is that the large shareholder wants to masquerade as a noise trader (that is, to become even more inconspicuous), and the noise trade has zero persistence. Thus, the amplification factor moves the fundamental in the direction of white noise. Note also that the amplification factor increases the level of the spectral density at every frequency—that is, the overall volatility of the fundamental is increased, as the analytical model demonstrates.

Figure 1. Plot of the spectral densities



This figure plots the spectral densities of the Φ process filter $(1-.93z)^{-1}$ (dashed line), the insider's amplification filter $(1-\Theta)$ (dotted line) and the net process filter (solid line). The amplification filter amplifies high frequencies more than low frequencies, thus reducing the persistence of the fundamental.

In the description of the static model, I noted that it would be possible to frame the penalty term in a more realistic way, that is, as a cost imposed on the large shareholder for failing to hit a target that represents the public objectives of the firm, thus capturing the trade-off between the public and private incentives of the large shareholder. In the dynamic model, there is an added dimension to this trade-off: the target might be a *process*, $T(L)\eta_t$. A Θ that fails to match the autoregressive structure of the target process will entail a penalty beyond the penalty stemming from the difference in magnitude: the Θ necessary to hit the target might have a very different structure from the Θ that moves the fundamental in the direction of resembling the noise trade process. This will further alter Θ and therefore move the private fundamental process away from the structure of the noise process.

One can look at it the other way: the visible target-hitting part of the large shareholder's behavior is going to be influenced by the "dark" unobservable side, so that the Θ process fails to directly offset the visible process $T(L)\eta(s)$. Thus, trading incentives that are hidden will distort the publicly observable allocation of resources of the firm.

7. Conclusion

The following phenomena were demonstrated:

(i) The large shareholder amplifies the unforecastable part of the fundamental in order to increase his trading profits.
(ii) In a dynamic setting, the informed trader behaves so as to be inconspicuous, so that total order flow has the autoregressive structure of the noise trade, and price has the autoregressive structure of the fundamental value process.
(iii) In a dynamic setting, the large shareholder alters the autoregressive structure of the firm's fundamental value process, that is, he dynamically obfuscates, and because this alteration is based on the market makers' forecast error process, it does not increase the amount of information available to the market.

Given that the large shareholder's trading profits can be enhanced by his alteration and amplification of the private signal, there is an incentive to acquire private information, along the lines set out in [7]. One usually unspoken element of private information models is the reason for the privacy or unobservability of the information. It is evident here that there are strong incentives to acquire private information and also to keep it private.

The amplification of this private information that takes place has a wider consequence. Recalling that Kyle's λ is a measure of price impact and is therefore inversely related to liquidity, and also that λ is positively related to the volatility of the fundamental value of the stock, a property that extends appropriately in the dynamic model as well, then the large shareholder's amplification of the volatility of the fundamental reduces market liquidity. The optimality of the resulting equilibrium level of liquidity is an open question.

In a more elaborate model the large shareholder would be enjoined to hit publicly observable fundamental targets. This would cause the large shareholder's activities to be divided between his fiduciary actions and the amplification of his privately observable shocks for trading purposes. The result is twofold. First, the autoregressive structure of the publicly observable shocks will cause the large shareholder's amplification of the private shock to be influenced by the autoregressive structure of the private shock and of the noise trade structure.

Second, the publicly observable actions will be influenced by the amplification effect, and in a dynamic setting this will mean that the publicly observable shocks will be imperfectly offset not only in their magnitude but in their time series structure as well—this, despite the fact that the public shocks, private shocks and noise trade shocks can all be mutually independent. One can view the influence of the private and noise trade shocks as a kind of "dark matter" that distorts the allocation of resources in the firm.

A central feature of business cycles is that they are persistent relative to the shocks that induce them. Successfully explaining this persistence requires explaining how firms fail to adjust quickly to shocks. The model here suggests that agents in possession of private information about fundamental shocks will not only obfuscate that information by amplifying the unforecastable part, they will add to the obfuscation by deliberately altering its autoregressive structure. Thus, if there is any uncertainty about aggregate nominal processes along the lines of [30], such obfuscation will actually exacerbate the associated signal extraction problem, and with it, deliver the aggregate fluctuations we observe.

Appendix A. On the equivalence of the direct and indirect methods of solving the static model

In this appendix I briefly recapitulate the demonstration of the equivalence of the direct and indirect approaches as outlined in Bernhardt and Taub [6].

Suppressing the effect of θ and taking the linear pricing rule as given, the large shareholder's optimization problem in direct form is

$$\max_x E\left[(e-\lambda(x+u))x\middle|e,p\right] \tag{58}$$

Observing price as well as the signal is equivalent to observing u as well; the problem is then deterministic, with solution

$$x=\frac{1}{2\lambda}e-\frac{u}{2}. \tag{59}$$

which is linear. It is useful to map this solution into the linear structure in equation (4) of the main text. After simple algebra,

$$b=\frac{1}{\lambda} \qquad \gamma=-\frac{1}{2}.$$

The market maker chooses the pricing rule to best predict the value e conditional on order flow due to the assumption that the market makers compete and earn zero profit. With the informed trader's order flow linear, λ is then calculated by a projection, which is

$$\lambda=\frac{\mathrm{cov}(e,x+u)}{\mathrm{var}(x+u)}=\frac{\mathrm{cov}(e,\frac{1}{2\lambda}e-\frac{u}{2}+u)}{\mathrm{var}(\frac{1}{2\lambda}e-\frac{u}{2}+u))}=\frac{\frac{1}{2\lambda}\sigma_e^2}{\frac{1}{4\lambda^2}\sigma_e^2+\frac{1}{4}\sigma_u^2}$$

Solving this equation for λ and substituting into the equation for b yields the solution in equation (13) of the main text.

The indirect approach begins with the assumption that the demand schedule is linear (as this has been justified by the solution in equation (59)). The optimization problem is then

$$\max_{\{b,\gamma\}} E\left[(e-\lambda(be+\gamma(be+u)+u))(be+\gamma(be+u))\middle|e,p\right] \tag{60}$$

Taking the expectation before calculating the first order conditions yields equation (7) in the main text, and the subsequent calculations yield the same results as the direct approach.

The dynamic version of the argument can be seen in for example Bernhardt, Seiler and Taub [9].

Appendix B. Remarks about Brownian processes in continuous time and their transforms

Technically speaking, the formulation in (21) is mathematically ill-defined: the sample paths of white noise in continuous time are not continuous or even measurable. To address this issue, Davis ([17] pp. 79-83) and Doob ([19] p. 426) begin their treatments of white noise by constructing Brownian motion via limiting arguments; for example, Davis develops Brownian motion as the sums of products of Haar functions (a type of step function) with Gaussian random variables. He later provides a second derivation using Ornstein-Uhlenbeck processes, which are well-defined via their correlation functions. Davis then examines the limiting properties of the Fourier transform of the Ornstein-Uhlenbeck covariance function. The limit of the Fourier transform is simply a constant, which is the Fourier transform of the delta function. The delta function is also an ill-posed object, but its integral is a step function, which is tractable. Davis then argues that Brownian motion can be well-defined as the integral of white noise, which by the limiting argument has a covariance function that is a step function, and is thus well posed. He concludes that integration of white noise, either in its pure form (yielding Brownian motion) or as convolutions with a filter such as ϕ (yielding a stationary process such as an Ornstein-Uhlenbeck process) is mathematically sound:

> The conclusion we arrive at from the above discussion is that we cannot represent mathematically white noise itself, but if it appears in integrated form then Brownian motion is an appropriate model. ([17] p. 82)

Doob and also Hansen and Sargent draw a similar conclusion.

As noted in the discussion of equation (25), the noise trade term $u(t)$ can be viewed as white noise in the limit as η goes to infinity. Given that white noise is itself ill-defined, it is mathematically more sound to view the Laplace transform of $u(t)$ as occurring before taking that limit, that is, it is the transform of the integrated process

$$u(t) \equiv \int_0^\infty e^{-\eta\tau} dN(t-\tau)$$

where $N(t-\tau)$ is a Brownian motion. If $\eta < \infty$, the Laplace transform of the integral is well-defined, and so one can think of it that way, taking the the limit of the Laplace transform as η tends to infinity. This is what Davis ([17] p. 80) does: specifically, he examines the Fourier transform of the (auto) covariance function of the Ornstein-Uhlenbeck process, that is,

$$\operatorname{cov}(u(t), u(s)) \equiv \sigma^2 e^{-\alpha|t-s|}$$

then lets the salient parameter (α in his notation) tend to infinity; this yields a flat spectrum which he notes is the transform of a delta function, which is the correlation function of white noise. Hansen and Sargent carry out a similar operation: see their first example ([24] p. 213).

As Kailath, Sayed and Hassibi and also Hansen and Sargent note, the Laplace transform is a special case of the Fourier transform, and care must be taken to ensure that the integration inherent in the transforms converges. Kailath, Sayed and Hassibi refer to exponential boundedness of the processes; this is a specialization of the more general requirement that the functions in question have no poles in the domain of interest. Specifically, as elaborated by Hansen and Sargent, I assume that functions are analytic in a strip along the imaginary line, and the imaginary line itself is in this strip because of discounting. Moreover the endogenous processes that will later be generated by optimization and equilibrium will automatically satisfy this criterion.

Appendix C. Fourier transforms of continuous-time processes

The Ornstein-Uhlenbeck process is the continuous-time analogue of the discrete autoregressive process:

$$dx = axdt + dz$$

The integral representation is

$$x(t) = \int_{-\infty}^{t} e^{a(t-s)} dz(t)$$

The Fourier transform of $e^{a(t-s)}$ is

$$\frac{1}{i\omega - a}$$

and the Fourier transform of dz (which corresponds to white noise) is the δ function. (A reference is Igloi and Terdik [26], p. 4.) The spectral density is

$$\frac{1}{a^2 + \omega^2}$$

The Fourier transform (and corresponding Laplace transform) resemble the pole forms $1/(z - a)$ in discrete time models. The building block in the s-domain is therefore also rational functions, except that causality is associated with poles in the left half plane instead of the unit circle. An additional reference is Hansen and Sargent [24].

Observe that $a = 1$ yields the Fourier transform of a standard Brownian motion; thus, with discounting it isn't a problem to translate standard discrete-time stationary models to this setting, and vice versa.

Appendix D. Practical details of spectral factorization and annihilator operations in continuous time

In this appendix I examine how factorization and the annihilation operator work in practical examples. I begin by briefly recapitulating an example from Kailath, Sayed and Hassibi, p. 263-264. Kailath, Sayed and Hassibi posit a model which has the following Wiener-Hopf equation:

$$K(s)S_y(s) = S_{sy}(s)e^{s\lambda} - G(s) \qquad \text{(KSH 7.A.4)}$$

Here $K(s)$ is the Laplace transform (s-transform) of the unknown filter that is to be found; $G(s)$ is the Laplace transform of a function $g(t)$ that is a purely anticausal function, that is, a function that is analytic on the left half plane only and zero in the right half plane, but which is otherwise arbitrary, corresponding to the principal part function $\sum_{-\infty}^{-1}$ in the discrete time setting: $g(t) = 0$, $t > 0$; $S_{sy}(s)$ and $S_y(s)$ are the Laplace transforms of variance and covariance functions

$$S_y(s) = \mathcal{L}\{R_y\} \qquad S_{ys}(s) = \mathcal{L}\{R_{ys}\}$$

with

$$R_y(\tau) \equiv E[\mathbf{y}(t)\mathbf{y}(t-\tau)] \qquad R_{ys}(\tau) \equiv E[\mathbf{s}(t)\mathbf{y}(t-\tau)]$$

Note that Kailath, Sayed and Hassibi have some contrasting notation: process $\mathbf{s(t)}$ is in boldface, and the argument of the Laplace-transformed function s, which is completely different. Thus, S_y is the Laplace transform of the observed process, and $\mathbf{s}(t)$ is the signal process that the observer wants to extract; R_{ys} is then the covariance function between the observed and signal processes.

The exponential term appears in the Wiener-Hopf equation because the original equation is shifted:

$$R_{sy}(t+\lambda) = \int_0^\infty k(\tau)R_y(t-\tau)d\tau, \qquad t > 0$$

which captures the idea of time-lagged observations.

To solve the problem ([27] 7.A.4), first factor S_y. Abstractly, this factorization is

$$\text{(KSH 7.A.2)} \qquad S_y(s) = L(s)RL^*(-s^*)$$

where R is a positive constant, and $L(s)$ is causal, that is, both L and L^{-1} are analytic on the right half plane.

Now write the solution:

$$\text{(KSH 7.A.7)} \qquad K(s) = L(s)^{-1}\left\{L^*(-s^*)^{-1}R^{-1}S_{xy}(s)e^{s\lambda}\right\}_+$$

The remaining agenda is to carry out a factorization for a practical problem and to demonstrate how the annihilation operation works in that practical setting.

Kailath, Sayed and Hassibi posit a signal process with Fourier transform spectral density

$$S_s(f) = \mathcal{F}\left\{e^{-\alpha|t|}\right\} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Note that there is a distinction between the Fourier and Laplace representations. Defining $s \equiv 2\pi i f$, the equivalent bilateral Laplace transform is

$$S_s(s) = \mathcal{L}\left\{e^{-\alpha|t|}\right\} = \frac{2\alpha}{\alpha^2 - s^2}$$

The noise process $\mathbf{v}(t)$ is white noise (not the same as a Brownian process!) which has a flat spectrum:

$$S_v(s) = 1$$

and the sum of the signal and noise, $\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{v}(t)$, is (because the Laplace transform of a sum is the sum of the Laplace transforms)

$$S_y(s) = S_s(s) + S_v(s) = \frac{2\alpha}{\alpha^2 - s^2} + 1 = \frac{2\alpha}{\alpha^2 - s^2} + 1 = \frac{s^2 - \alpha^2 - 2\alpha}{s^2 - \alpha^2} = L(s)RL^*(-s^*).$$

The denominator of this expression is the product $(s-\alpha)(s+\alpha)$. Restate the entire expression as a product:

$$\frac{s+\sqrt{\alpha^2+2\alpha}}{s+\alpha}\frac{s-\sqrt{\alpha^2-2\alpha}}{s-\alpha}$$

so that

$$L(s)=\frac{s+\sqrt{\alpha^2+2\alpha}}{s+\alpha}.$$

(Of course this is just one of the potential factorizations.) Note that L is analytic in the right half plane because its pole, $-\alpha$, is in the left half plane, and the inverse is analytic in the right half plane because the zero, $-\sqrt{\alpha^2+2\alpha}$, is in the left half plane.

The final step is to calculate the annihilate. To do this, a partial fractions calculation must be done. The argument of the annihilator is

$$\frac{s-\alpha}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{\alpha^2-s^2}$$

Writing out the factors in the denominator, there is a cancellation:

$$=\frac{s-\alpha}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{(\alpha-s)(\alpha+s)}=-\frac{1}{s-\sqrt{\alpha^2+2\alpha}}\frac{2\alpha}{\alpha+s}.$$

Now rewrite this with partial fractions:

$$=\frac{-\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s-\sqrt{\alpha^2+2\alpha}}+\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}.$$

The annihilator kills elements that have poles in the right half plane; the first term will therefore be killed:

$$\left\{\frac{-\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s-\sqrt{\alpha^2+2\alpha}}+\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}\right\}_+=\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}.$$

Therefore the solution of the Wiener-Hopf equation is

$$K(s)=\frac{s+\alpha}{s+\sqrt{\alpha^2+2\alpha}}\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{\alpha+s}=\frac{\frac{2\alpha}{\alpha+\sqrt{\alpha^2+2\alpha}}}{s+\sqrt{\alpha^2+2\alpha}}.$$

This is the Laplace transform for a filter. The actual filter can be obtained by performing the inverse transform operation.

D.1. **A small lemma about the annihilator.** The annihilator operator is a linear operator and therefore can be expressed as an integral ([27], p. 263):

$$\{F(s)\}_+=\int_0^\infty\left[\frac{1}{2\pi i}\int F(p)e^{pt}dp\right]e^{-st}dt \tag{61}$$

The interpretation is straightforward: perform the inverse Laplace transform with the inner integral (which in conventional situations is integrated along the imaginary axis). Then perform the one-sided Laplace transform on the result, which picks up only the part of the function defined for positive t, that is, in the right half plane. The following small lemma holds, which is a variation of Whittle's theorem.

Lemma 15. *Let F be analytic in the right half plane, and let $a > 0$. Then*

$$\left\{F^*(r - s^*)\frac{1}{s + a}\right\}_+ = F(r + a)\frac{1}{s + a}$$

Proof: I will first demonstrate this for a simple version of F, namely $F(s) = \frac{1}{s+b}$, $b > 0$—namely when F is also the filter for an Ornstein-Uhlenbeck process. In that case, the inner integral of (61) is

$$\frac{1}{2\pi i}\int \frac{1}{-p + r + b}\frac{1}{p + a}e^{pt}dp$$

Now do partial fractions:

$$= \frac{1}{2\pi i}\int \left(\frac{\frac{1}{r+b+a}}{-p + r + b} + \frac{\frac{1}{r+b+a}}{p + a}\right)e^{pt}dp$$

The integration is along the imaginary axis. This is equivalent (via a Möbius transform) to integrating around the unit circle. Consequently Cauchy's theorem can be invoked: a holomorphic function with a pole in the right half plane integrates to zero. The pole of the first term in the expression is $p + r$, and therefore the integral of the first term is zero. The remaining expression is

$$\frac{1}{r + b + a}e^{-at}$$

Now take the outer integral.

$$\int_0^\infty \left[\frac{1}{r + b + a}e^{-at}\right] e^{-st}dt = \frac{1}{r + b + a}\frac{1}{s + a}.$$

This completes the proof for this simple case.

If f is analytic, then it can be represented in power series form:

$$f(\tau) = \sum_{k=0}^{\infty} f_k e^{-b_k\tau}.$$

The s-transform of this function is

$$F(s) = \sum_{k=0}^{\infty} f_k\frac{1}{s + b_k}.$$

Now proceed as in the proof above for each k. □

This result is stated and proved in greater generality for matrix systems in Seiler and Taub [33], Lemma C.18, using state space methods. When general compound expressions of the sort $\{FG^*\}_+$, where both F and G are analytic, that is, their poles are in the left half plane, are viewed from a state space perspective, it is clear that the product will be a function with poles in the left half place inherited from the poles of F and poles in the right half plane inherited from G. The annihilator removes the latter poles, while the poles of F survive.

Appendix E. Some propositions about the order flow and price processes in equilibrium: Inconspicuousness

As also demonstrated in [9], [8] and [33], the forecast error characterization of the trading strategies has broader implications. First, because the informed traders do not want to be detected by the market makers or by their rivals, they hide behind the noise traders. This requires that the order flow process have no dynamic structure that would allow market makers to infer the informed trades. Therefore, the total order flow will have the same stochastic structure as the noise trade process.

Second, the price process must not have a dynamic structure that is fundamentally different from the dynamic structure of the fundamental asset value process, as this would allow the informed traders to arbitrage against it purely based on filtering the dynamic structure.

These results are general in that they hold for multiple informed traders. In this appendix the vector notation reflects this generality.

Proposition 16. *The total order flow process filter*

$$JH$$

is a constant matrix. Therefore order flow has the same dynamic structure as the filter for the noise trade process $u(t)$.

Proof: Add up the Γ_i equations (51) across traders, yielding

$$J^*J\sum_i \Gamma_i = -B^*R_e\sum_i \begin{pmatrix} 0 \\ \vdots \\ B_i \\ \vdots \\ 0 \end{pmatrix} + \mathcal{A}^*$$

Recalling the definition $H \equiv \sum_i \Gamma_i + I$ and using the vector expression B,

$$J^*J(H-I) = -B^*R_eB + \mathcal{A}^*.$$

From the definition of J in equation (47),

$$J^*J(H-I) = -(J^*J - R_u) + \mathcal{A}^*$$

where R_u is the covariance function for the noise trade process. Also recall that it is assumed that the covariance functions for the fundamental processes are Dirac δ functions, so R_u is a constant matrix. Because no further filter is applied to the noise trade process, noise trade is white noise.

Algebraic manipulation then yields

$$J^*JH = R_u + \mathcal{A}^*.$$

Order flow is then

$$JH = \left\{ J^{*-1} R_u \right\}_+$$

Because R_u is a constant matrix, the projection operator eliminates all poles in the negative half plane. But J was constructed via factorization to have poles only in the positive half plane, and therefore the projection must be a constant. □

Proposition 17. *The price process filter*

$$JH\Lambda$$

has the same pole structure as the value process filter Φ*, and therefore the price process has the same dynamic structure as the value process.*

Proof: The filter for the price process is the order-flow filter JH multiplied by the pricing filter Λ.

Multiplying the first-order condition for Λ, (48) by J^{*-1} and applying the annihilator yields the equation for the price process,

$$JH\Lambda = \left\{ J^{*-1} B^* R_e \Phi \right\}_+ .$$

By Lemma 15 in Appendix D, the right hand side is the product of a constant matrix and Φ. Thus,

$$JH\Lambda = C R_e \Phi. \tag{62}$$

where C is a constant matrix $J(r+\rho)^{-1} B(r+\rho)$. □

It should be noted that if the large shareholder chooses Θ, then the proposition holds for $\tilde{\Phi} = (I - \Theta)\Phi$, where Φ is the filter for the unmodified fundamental process. The application of Lemma 15 then requires the matrix formulation in which the non-pole part of $\tilde{\Phi}$ is represented by a pole at infinity.

E.1. **Acceleration.** As was shown in [33] and [9], the informed traders trade intensely on their information, in the sense that the filter on the fundamentals of their private signals has a pole structure such that their order flow on private signals is less serially correlated than the asset value itself. The proof of this was set out in [33] for the discrete time multi-asset case. The proof there has two main parts. The first part is to establish that the poles exceed the poles of Φ. This is done by showing that the equilibrium mapping of a conjectured pole structure for b results in a set of larger poles. The second part follows by showing that the number of poles increases by one for each iteration of the mapping, and that therefore there must be infinitely many in equilibrium.[25]

[25] To clarify terminology, a pole can be intuitively viewed as the inverse of an autoregressive coefficient in discrete time, and is therefore (for stationary processes in the discrete-time setting of [33]) greater than the square root of the discount factor in absolute value. In continuous time poles correspond to points in the right half plane. As the number of poles is infinite, this requires that the poles converge to infinity. It can then

E.2. **Discussion and related literature.** The results here are a little more general than some of the literature in the following sense. In Danilova [15] for example, which has dynamic evolution of the fundamental value and a single informed trader, the hiding idea is termed *inconspicuousness*. But in that model, the asset fundamental value process and the noise trade are both Brownian processes with jumps, so the dynamic structure or order flow is no different than the price process. Here, by contrast, the noise trade is serially uncorrelated while the fundamental value process is serially correlated. Therefore in order to hide, the informed traders must adjust not just the magnitudes of their trades, but their dynamic pattern.

E.2.1. *Relationship with the order-splitting literature.* The reason for the order flow result is related to the result of Back and Baruch [4]: the informed traders pool with the noise traders, that is, they hide their trades. The Back and Baruch model establishes that breaking up large block orders into a sequence of small ones is optimal, but the result here emphasizes that it is not the breaking up of the orders that is crucial, but the fact that the orders are stochastically indistinguishable from the noise trades that matters.

In this sense the model also suggests that there is not an important difference between dealership markets and other market structures such as a limit order market, buttressing Back and Baruch's central finding.

Back and Baruch [4] set out a model in which informed traders can post orders of any size; in equilibrium they order one share at a time, with large (block) orders being expressed as a high rate of single-share orders. The result is that informed traders pool with uninformed traders. The results here are equivalent: informed traders want to appear like noise traders, otherwise their information can be extracted by market makers. Back and Baruch demonstrate the equivalence of their market structure with one in which there is an open (public) order book with limit orders, in which informed traders put in limit orders.

In particular, Back and Baruch note that in a floor-trading model—that is, one with competitive market makers, as in the Kyle [28] model, the informed traders might submit a large order, but it must be structured (via a mixed strategy) so that market makers cannot clearly identify it as an informed order as would be the case in a separating equilibrium:

> When orders are worked, liquidity providers on a floor exchange can of course condition on the size of an order, but they cannot condition on the size of the demand underlying the order—they cannot know whether there will be more orders from the same trader in the same direction immediately forthcoming. Thus, in a pooling equilibrium on a floor exchange, ask prices are upper-tail expectations—expectations conditional on the size of the demand being the size of the

be argued that arbitrary rational functions can be approximated by the sums of such pole terms, and also characterized by the pattern of weights on those pole terms.

> order or larger—precisely as in a limit-order market. This is the reason a pooling (worked-order) equilibrium on a floor exchange is equivalent to a block-order equilibrium in a limit-order market. ([4], p. 2)

E.2.2. *Relationship to the speed of revelation literature.* The high intensity of trading relative to information arrival result matches the similar result from the discrete-time models in Bernhardt, Seiler and Taub [9] and [33], also expressed by the infinitely long pattern of poles. There is a related result in the paper of Chau and Vayanos [13]. The Chau and Vayanos model uses a different information structure: market makers can observe firm value contemporaneously, but cannot forecast its evolution, while the informed trader can forecast. In their model, which has a single informed trader in an infinite horizon setting with stationary asset value evolution and continuous information arrival, trading intensity is accelerated relative to the arrival rate of information, but because of the model structure the acceleration results in full and immediate revelation of the informed trader's information.

Appendix F. Factoring the coefficient matrix

Combining the first order conditions for B and Θ in equations (43) and (54) yields the vector condition,

$$\begin{pmatrix} H\Lambda H^* + H\Lambda^* H^* & \Phi H^* \\ H\Phi^* & C\Phi\Phi^* \end{pmatrix} \begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} \Phi H^* \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{63}$$

The coefficient matrix on the left is Hermitian and so can be factored. The coefficient matrix can be written as

$$\begin{pmatrix} H^* & 0 \\ 0 & \Phi^* \end{pmatrix} \begin{pmatrix} H^{-1}H\Lambda + \Lambda^* H^* H^{*-1} & 1 \\ 1 & C \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & \Phi \end{pmatrix} \tag{64}$$

The internal matrix is then easier to factor because only the upper left element is nonscalar.

From equation (64) we have:

$$\begin{pmatrix} H^* & 0 \\ 0 & \Phi^* \end{pmatrix} \begin{pmatrix} \Lambda + \Lambda^* & 1 \\ 1 & C \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} \Phi H^* \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{65}$$

and then use the factorization of the inner part:

$$\begin{pmatrix} H^* & 0 \\ 0 & \Phi^* \end{pmatrix} \bar{F}^* \bar{F} \begin{pmatrix} H & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} \Phi H^* \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{66}$$

We can invert the outer parts and then the inner parts:

$$\bar{F} \begin{pmatrix} H & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} B \\ \Theta \end{pmatrix} = \bar{F}^{*-1} \begin{pmatrix} \Phi \\ 0 \end{pmatrix} + \mathcal{A}^* \tag{67}$$

The cancellation of the H^* on the right assumes that this is a scalar and invertible quantity. The solution is

$$\begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & \Phi \end{pmatrix}^{-1} \bar{F}^{-1} \left\{ \bar{F}^{*-1} \begin{pmatrix} \Phi \\ 0 \end{pmatrix} \right\}_+$$

Now recall from Proposition 16 that $1 + \Gamma = H \sim J^{-1}$. Thus,

$$\begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \bar{F}^{-1} \left\{ \bar{F}^{*-1} \begin{pmatrix} \Phi \\ 0 \end{pmatrix} \right\}_+$$

where c_2 is a constant that can be derived from Proposition 16. Taking this a step further we have

$$\begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \bar{F}^{-1} \left\{ \begin{pmatrix} (\bar{F}^{*-1})_{11} \Phi \\ (\bar{F}^{*-1})_{21} \Phi \end{pmatrix} \right\}_+$$

Recalling that $\Phi(s)$ is Ornstein-Uhlenbeck (the continuous time analogue of autoregressive) $\frac{1}{s+a}$, and invoking the annihilator theorem (Lemma 15 in Appendix D)

$$\begin{pmatrix} B \\ \Theta \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \bar{F}^{-1} \begin{pmatrix} (\bar{F}^{-1})_{11} (r+a) \Phi \\ (\bar{F}^{-1})_{12} (r+a) \Phi \end{pmatrix} \tag{68}$$

When the algebra is carried further, the Φ terms will cancel from the solution for Θ. Only the factor $\bar{F}$ then influences Θ directly. The exact structure of $\bar{F}$ can be used to show that Θ is not a constant.

This result can now be used to numerically calculate F and the inverses, products and annihilates.

Appendix G. Extending the Cholesky decomposition to matrices of rational functions

In this appendix I detail how to extend the Cholesky decomposition from ordinary matrices to rational functions. The agenda is to develop the candidate factor, but not require that the candidate factor be analytic and invertible.

One begins with a Hermitian $n \times n$ matrix H, with elements h_{ij}, that is to be factored. The immediate question is whether to right- or left-factor H:

$$H = LL^* \quad \text{left factor} \qquad \text{or} \qquad H = R^*R \quad \text{right factor}$$

I will follow the left factor strategy. This is sufficient, even if we want a right factorization: first observe that if H is Hermitian, then so is the transpose H'. Thus,

$$H' = (LL^*)' = (L^*)'L'$$

which is a right factorization of H', and

$$H' = (R^*R)' = R'(R^*)'.$$

which is a left factorization. Thus, to obtain a right factorization of H, we just need to find a left factorization of H' and take the transpose of the result R'. Thus, it is sufficient to consider the left factorization; the right factor is simply the transpose of the left factor of H'.

Next, begin the factorization. Note: this is highly parallel with the development on the Wikipedia page on the Cholesky decomposition for ordinary matrices. The algorithm is recursive. In the first step, we have

$$H^1 \equiv H.$$

At step i of the algorithm there is an intermediate matrix,

$$H^i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & a_{ii} & b_i^* \\ 0 & b & b \end{pmatrix}$$

where I_i is the i-dimensional identity, a_{ii} is the ith diagonal entry from H_i, b is the $(n-i) \times 1$ column vector and the block matrix b is the lower right $(n-i) \times (n-i)$ submatrix from H.

When H is a matrix of numbers, then we take the square root of a_{ii}, and construct the matrix

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \sqrt{a_{ii}} & 0 \\ 0 & \frac{1}{\sqrt{a_{ii}}} b & I_{n-i} \end{pmatrix}$$

The H_2 operation equivalent to the square root is spectral factorization. Thus, we want to find f_i such that

$$a_{ii} = f_i^* f_i$$

This is relatively straightforward because by construction a_{ii} is a scalar function. Thus in the spectral factorization case,

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i & 0 \\ 0 & f_i^{*-1} b & I_{n-i} \end{pmatrix}$$

Now define

$$H_{i+1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b - a_i^{-1} b b^* \end{pmatrix}$$

and

$$H_{n+1} = I_n$$

Then

$$L = L_1 L_2 \cdots L_n.$$

There is one detail: in the H_2 case, how do we know that

$$L_i = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i & 0 \\ 0 & f_i^{*-1} b & I_{n-i} \end{pmatrix} \qquad \text{not} \qquad \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & f_i^* & 0 \\ 0 & f_i^{-1} b & I_{n-i} \end{pmatrix} \qquad ?$$

In the $n = 2$ case, we can do the multiplication. First,

$$H_1 = H = \begin{pmatrix} h_{11} & h_{21}^* \\ h_{21} & h_{22} \end{pmatrix},$$

$$L_1 = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & h_{22} - h_{11}^{-1} h_{21} h_{21}^* \end{pmatrix}$$

with

$$f_1 f_1^* = h_{11}$$

Defining g by the factorization

$$g g^* \equiv h_{22} - h_{11}^{-1} h_{21} h_{21}^*.$$

Then

$$L_2 = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$$

The factor is then

$$L = L_1 L_2 = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix}$$

so

$$\begin{aligned} H = LL^* &= \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix}^* \\ &= \begin{pmatrix} f_1 & 0 \\ f_1^{*-1} h_{21} & g \end{pmatrix} \begin{pmatrix} f_1^* & h_{21}^* f_1^{-1} \\ 0 & g^* \end{pmatrix} \\ &= \begin{pmatrix} f_1 f_1^* & h_{21}^* \\ h_{21} & h_{21}^* f_1^{-1} f_1^{*-1} h_{21} + g g^* \end{pmatrix} \\ &= \begin{pmatrix} h_{11} & h_{21}^* \\ h_{21} & h_{22} \end{pmatrix} \end{aligned}$$

Once this initial factor L is constructed, the Ball-Taub algorithm [2] can be applied to convert the factor to analytic and invertible form. That algorithm is presented under that assumption that a preliminary *right* factor R has been found and the appropriate adjustment must be made.

Appendix H. Factoring an example

Let us consider a simple example in which the fundamental firm value process is an AR(1) process and expressed by

$$\Phi(z) = \frac{1}{1 - az}.$$

We know from previous reasoning that this will lead to the pricing filter to have the same structure as the fundamental process, that is,

$$\Lambda = \frac{\lambda}{1 - az}.$$

For that reason, the matrix to be factored is

$$H \equiv \bar{F}^* \bar{F} = \begin{pmatrix} \Lambda + \Lambda^* & 1 \\ 1 & C \end{pmatrix}$$

In the following example, $a = .93$ (so $a^{-1} = 1.075$) and $C = 1.16$. We want to find a right factor, but the algorithm is set up to find a left factor; we therefore take the transpose of H and find the left factor, then the transpose of that factor. However, because of the symmetry of the H we have $H' = H$.

Putting the numbers in and performing an initial factorization yields the left factor

$$H = \begin{pmatrix} \frac{(z-1.470)(z-.680)}{(z-1.075)(z-.930)} & 1 \\ 1 & 1.16 \end{pmatrix}$$

Observe that the outside zero of the numerator in the $(1,1)$ element, which characterize the AR part of the implied process, is larger than thee denominator zero (1.075), so the persistence arising from the MA part of the implied process will be lower than that induced by the AR part. The invertible left factor of H, which is simply the initial candidate factor generated by the Cholesky factorization, is

$$\begin{pmatrix} \frac{.855(z-1.470)}{z-1.075} & 0 \\ \frac{.855(z-.93)}{z-.680} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

Multiply by a Blaschke factor from the right:

$$\begin{pmatrix} \frac{.855(z-1.470)}{z-1.075} & 0 \\ \frac{.855(z-.93)}{z-.680} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} \begin{pmatrix} -\frac{z-.680}{1-.680z} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -\frac{.855(z-1.470)}{z-1.075}\frac{z-.680}{1-.680z} & 0 \\ -\frac{.855(z-.93)}{1-.680z} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

and with cancellation in the upper left element,

$$\begin{pmatrix} \frac{\frac{.855}{.680}(z-.680)}{z-1.075} & 0 \\ \frac{\frac{.855}{.680}(z-.93)}{z-1.47} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} = \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & 0 \\ \frac{1.257(z-.93)}{z-1.47} & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

which has a zero but no poles. The transpose of this, which is the right factor is,

$$\bar{F} = \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & \frac{1.257(z-.93)}{z-1.47} \\ 0 & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix}$$

Because of the zero, additional factorization is needed. First, we calculate the constituent elements of the Θ matrix (the coefficient matrix from [2]):

$$A = \begin{pmatrix} .68 \end{pmatrix} \qquad B = \begin{pmatrix} -.884 & -.468 \end{pmatrix} \qquad \Omega = \begin{pmatrix} 1.86 \end{pmatrix}$$

yielding

$$\Theta = \begin{pmatrix} -\frac{.930(z-.494)}{z-1.470} & -\frac{1.022(z-1.0)}{z-1.470} \\ -\frac{1.022(z-1.0)}{z-1.470} & \frac{.459(z-2.025)}{z-1.470} \end{pmatrix}$$

The invertible factor is

$$\Theta^*\bar{F} = \begin{pmatrix} -\frac{.312(z-2.025)}{z-.68} & -\frac{.695(z-1.0)}{z-.68} \\ -\frac{.695(z-1.0)}{z-.68} & \frac{.632(z-.494)}{z-.68} \end{pmatrix} \begin{pmatrix} \frac{1.257(z-.680)}{z-1.075} & \frac{1.257(z-.93)}{z-1.47} \\ 0 & \frac{.296(z-2.686)}{z-1.470} \end{pmatrix} = \begin{pmatrix} \frac{.393(z-2.025)}{z-1.075} & .187 \\ \frac{.874(z-1.0)}{z-1.075} & 1.061 \end{pmatrix}$$

The inverse is

$$\bar{F}^{-1} = \begin{pmatrix} \frac{4.199(z-1.075)}{z-1.075} & -\frac{.739(z-1.075)}{z-2.686} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix}$$

We can write this as

$$\bar{F}^{-1} = \begin{pmatrix} \frac{4.199}{z-1.075}\Phi^{-1} & -\frac{.739}{z-2.686}\Phi^{-1} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix}$$

Inserting this into (68) yields

$$\begin{pmatrix} b \\ \theta \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \begin{pmatrix} \frac{4.199}{z-1.075}\Phi^{-1} & -\frac{.739}{z-2.686}\Phi^{-1} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix} \begin{pmatrix} \bar{F}_{11}(r+a)^{-1}\Phi \\ \bar{F}_{12}(r+a)^{-1}\Phi \end{pmatrix} \tag{69}$$

Clearly there will be several cancellations of the Φs. This yields

$$\begin{pmatrix} b \\ \theta \end{pmatrix} = \begin{pmatrix} c_2 J & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{4.199}{z-1.075} & -\frac{.739}{z-2.686} \\ -\frac{3.452(z-1)}{z-2.686} & \frac{1.552(z-2.025)}{z-2.686} \end{pmatrix} \begin{pmatrix} \bar{F}_{11}(r+a)^{-1} \\ \bar{F}_{12}(r+a)^{-1} \end{pmatrix} \tag{70}$$

Remembering that $\bar{F}_{11}(r+a)^{-1}$ and $\bar{F}_{12}(r+a)^{-1}$ are constants, it is evident that θ (the obfuscation coeffient) is not a constant:

$$\theta = -\bar{F}_{11}^{-1}(r+a)\frac{3.452(z-1)}{z-2.686} + \bar{F}_{12}^{-1}(r+a)\frac{1.552(z-2.025)}{z-2.686}$$

which is an ARMA(1,1) transfer function. Thus, the large shareholder adds persistence (mainly through the AR part) to the fundamental process.

We can be more explicit about the solution using the annihilator lemma (using discrete time):

$$\bar{F}^{*-1}\Big|_{z=a} = \begin{pmatrix} \frac{4.199(.93-1.075)}{.93-1.075} & -\frac{3.452(.93-1)}{.93-2.686} \\ -\frac{.739(.93-1.075)}{.93-2.686} & \frac{1.552(.93-2.025)}{.93-2.686} \end{pmatrix} = \begin{pmatrix} .347 & -.138 \\ -.0611 & .967 \end{pmatrix}$$

Our equation is then

$$\theta = -.347\frac{3.452(z-1)}{z-2.686} - .138\frac{1.552(z-2.025)}{z-2.686} = -\frac{1.41(z-1.155)}{z-2.686}$$

which is an ARMA(1,1) filter. Moreover, it does not cancel with Φ, and so the effective fundamental process will be

$$(1-\theta)\Phi = \frac{2.410(z-1.790)}{z-2.686}\Phi.$$

A slight variant of this example is discussed in the main text. [Figure 1 is for a slightly different case: the lower right element of the matrix to be factored is 1.1 rather than 1.18.] As the adjustment cost is increased (not shown), the amplification filter flattens out and shrinks.

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